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A Multi-Level Fractional Programming Problem with Stochastic Parameters in Constraint

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***Abstract* – This paper proposes an approach where it can be applied to the optimization decision making problems under various uncertainties and solves a multi-level fractional programming problem (MLFPP) involving stochastic parameters coefficient in which some dependent variables are to be constrained with a predefined probability. Such problems are called optimization under chance constraints. In this work, the first phase of the solution approach, we convert the probabilistic nature (stochastic) of this problem in the constraints into a multi-level fractional programming problem (MLFPP). Then the second phase, we use the 1Storder Taylor series polynomial series to convert (MLFPP) into a multi-level linear programming problem (MLFPP) for generating a compromise solution for this problem. In addition, a numerical example is provided to demonstrate the correctness of the proposed solution.**
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***Keywords* – Multi-Level Programming, Fractional Programming, Stochastic Programming, Chance Constraints.**

I. Introduction

Multilevel optimization problems have attracted considerable attention from the scientific and economic community in recent years. The multilevel system has extensive existences in management and decision making fields. Usually, this kind of problems can be solved by using different mathematical programming techniques [1, 2, 3, and 11].

Fractional programming problem is that in which the objective function is the ratio of numerator and denominator. These types of problems have attracted considerable research and interest. Fractional programming is useful in production planning, financial and corporate planning, health care and hospital planning etc. [2, 7, and 12].

Stochastic or chance constrained programming is an effective and convenient approach to control risk in decision making under uncertainty. However, due to unknown probability distributions of random parameters, the solution obtained from a chance constrained optimization problem can be biased. In practice, instead of knowing the true distribution of a random parameter, only a series of historical data, which can be considered as samples taken from the true (while ambiguous) distribution, can be observed and stored [4, 8, 9, and 10].

In literature, most studies have focused to solve chance constrained multi-objective fractional programming problem [2, 5, 6,7, 8, and 9]. In [2], Emam et al.suggested an approach to convert the stochastic multi-level large scale fractional programming problem (SMLLSFPP) into a multi-level large scale fractional programming problem then using Taylor series avoid the complexity of fractional nature.

In [5], a qualitative analysis of some stability notions of multi objective integer linear fractional programming problem under randomness in the right-hand side of the constraints only and that the random variables were distributed. Some of these notions were defined and characterized.

A parametric study on stochastic multi objective integer linear programming problems was presented in [6] and the stability of efficient solutions for such problems has been investigated. The chance constrained bi-level integer linear fractional programming problem has been solved by using Taylor series approach combined with the cutting-plane algorithm in [7].

In [8], Saad and farag proposed a solution for solving chance-constrained multi objective integer quadratic programming problem with some stability notions

In [9], Saad, et al. presented a Taylor series for solving chance constrained multi objective integer linear fractional programming problem.

This paper is organized as follows: we start in Section 2 by formulating the model of amulti-level fractional programming problem with stochastic parameters in constraint along with the solution concept. Section 3, transforming stochastic parameters in constraint into multi-level fractional programming problem is presented. In Section 4, using the Taylor series approach for to solve multi-level fractional programming problem is described. In addition, a numerical example is provided to illustrate the developed results in Section 5. Finally, conclusion and future works are reported in Section 6.

II. Problem Formulation and Solution Concept

The multi-level fractional programming problem with stochastic parameters in constraints can be formulated as follows:

[FLDM]

$$\underset{x_1}{Max} \quad F_1(x) = \frac{c_{1i}^T x + \alpha_{1i}}{d_{1i}^T x + \beta_1}, (i = 1,2,..., N_1), \tag{1}$$

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Where x_2, x_3 solve

[SLDM]

$$\underset{x_2}{Max}\ F_2(x) = \frac{c_{2j}^T x + \alpha_{2j}}{d_{2j}^T x + \beta_2}, (j = 1,2,..., N_2), \tag{2}$$

Where x_3 solves

[TLDM]

$$\underset{x_3}{Max}\ F_3(x) = \frac{c_{3r}^T x + \alpha_{3r}}{d_{3r}^T x + \beta_3}, (r = 1,2,..., N_3), \tag{3}$$

Subject to

$$x \in [M]. \tag{4}$$

Where

$$M = \{x \in R^n \mid P\{g_i(x) = \sum_{j=1}^{n} \alpha_{ij} x_j \le b_i\} \ge \alpha_i, \tag{5}$$

$$(x_j \ge 0), (i = 1,2,..., m), (j = 1,2,..., n)\}$$

In the above problem (1)-(5), $x_i \in R^n, (i = 1,2,3)$ be a vector variable indicating the first decision level's choice, the second decision level's choice and the third decision level's choice. $F_i : R^n \to R^{N_i}, (i = 1,2,3)$ be the first level objective function (FLMD), the second level objective function (SLDM) and the third level objective function (TLDM) have N_1, N_2 and N_3 objective functions, respectively. M is the feasible set choices $\{(x_1, x_2, x_3)\}$.

Furthermore, P means probability and α_i is a specified probability value. This means that the linear constraints may be violated some of the time and at most $100\ (1 - \alpha_i)\%$ of the time.

For the sake of simplicity, we assume that the random parameters $b_i, (i = 1,2,..., m)$ are distributed normally with known means $E\{b_i\}$ and variances $Var\{b_i\}$ and independently of each other.

The concept of the efficient solution of a multi-level fractional programming problem with stochastic parameters in constraint is introduced in the following definition.

Definition 1.

Let M_1, M_2, M_3 be the feasible regions of FLDM, SLDM and TLDM, respectively. For any $x_1 \in M_1 = \{x_1 | (x_1, x_2, x_3) \in M_1\}$ given by FLDM, and $x_2 \in M_2 = \{x_2 | (x_1, x_2, x_3) \in M_2\}$ given by SLDM, if the decision-making variable $x_3 \in M_3 = \{x_3 | (x_1, x_2, x_3) \in M_3\}$ is the optimal solution of the TLDM, then (x_1, x_2, x_3) is a feasible solution of a multi-level fractional programming problem with stochastic parameters in constraint.

Definition 2.

If (x_1^*, x_2^*, x_3^*) is a feasible solution of a multi-level fractional programming problem with stochastic parameters in constraint (1)-(5) with probability in constraints; no other feasible solution $(x_1, x_2, x_3) \in M$ exists, such that $f_{1i}(x_1^*, x_2^*, x_3^*) \le f_{1i}(x_1, x_2, x_3)$, with at least one $(i = 1, 2, ..., k_1)$; so (x_1^*, x_2^*, x_3^*) is the optimal solution of this problem.

III. Chance Constrained Transformation

In this section, we will convert the multi level fractional programming problem with stochastic parameters in constraints. The basic idea in treating of this problem is to convert the probabilistic nature (probability) into an equivalent deterministic form. In this case, the set of constraints can be rewritten by using the interesting technique of chance-constrained programming [7] as follows:

$$M' = \{x \in R^n \mid \sum_{j=1}^{n} \alpha_{ij} x_j \le E\{b_i\} + K_\alpha \sqrt{Var\{b_i\}}, \tag{6}$$

$$(x_j \ge 0), (i = 1,2,..., m), (j = 1,2,..., n)\}.$$

Where K_{α_i} is the standard normal value such that $\phi(K_{\alpha_i}) = 1 - \alpha_i$; and $\phi(a)$ represents the "cumulative distribution function" of the standard normal distribution evaluated at a.

Therefore, the (MLFPP) equivalent to a multi-level fractional programming problem with stochastic parameters in may be formulated as follows:

[FLDM]

$$\underset{x_1}{Max}\ F_1(x) = \frac{c_{1i}^T x + \alpha_{1i}}{d_{1i}^T x + \beta_1}, (i = 1,2,..., N_1), \tag{7}$$

Where x_2, x_3 solve

[SLDM]

$$\underset{x_2}{Max}\ F_2(x) = \frac{c_{2j}^T x + \alpha_{2j}}{d_{2j}^T x + \beta_2}, (j = 1,2,..., N_2), \tag{8}$$

Where x_3 solves

[TLDM]

$$\underset{x_3}{Max}\ F_3(x) = \frac{c_{3r}^T x + \alpha_{3r}}{d_{3r}^T x + \beta_3}, (r = 1,2,..., N_3), \tag{9}$$

Subject to

$$x \in M'. \tag{10}$$

IV. Taylor Series Approach for Multi-Level Fractional Programming Problem

In the deterministic multi-level fractional programming problem (MLFPP), we can be transform objective functions by using 1^{st} order Taylor series into (MLLPP) for the FLDM, SLDM and TLDM. Then solved linear programming by simplex method and putting the values of x_1, x_2 and x_3 in the objective functions, we can obtain the optimal solutions.

The proposed approach to solve a multi-level fractional programming problem (MLLFPP) can be explained as follows:

Step 1: Determine where $x_i^* = (x_{i1}^*,..., x_{im}^*)$ which is the value that is used to maximized the i^{th} objective function $F_i(x)$, $(i = 1,2,..., m)$ where m is a number of the variables.

Step 2: Transform the objective functions $F_i(x)$, $(i = 1,2,..., m)$ by using the 1^{st} order Taylor series polynomial series in the following form stated in ([7], [9]) as:

$$F_i(x) \cong \overline{F}_i(x) = F_i(x_i^*) + \sum_{j=1}^{n}(x_j - x_{ij}^*)\frac{\partial F_i(x_i^*)}{dx_j}, (j = 1,2,3) \quad (11)$$

Step 3: Using simplex method to solve (MLLPP). Then put the values of x_1, x_2 and x_3 in the objective functions to obtain the solutions.

V. An Illustrative Example

To demonstrate the solution of a multi-level fractional programming problem with stochastic parameters in constraint, let us consider the following example:

[FLDM]

$$\underset{x_1}{Max}\ F_1(x) = \frac{6x_1 + 3x_2 + 2x_3}{x_1 + x_2 + x_3 + 4},$$

Where x_2, x_3 solve

[SLDM]

$$\underset{x_2}{Max}\ F_2(x) = \frac{4x_1 + 5x_2 + 2x_3}{x_1 + x_2 + x_3 + 5},$$

Where x_3 solves

[TLDM]

$$\underset{x_3}{Max}\ F_3(x) = \frac{2x_1 + 3x_2 + 5x_3}{x_1 + x_2 + x_3 + 3},$$

Subject to

$$M = \{x \in R^3 : (x_1, x_2, x_3) \mid$$
$$\Pr\{x_1 + x_2 + x_3 \le b_i\} \ge 0.92,$$
$$\Pr\{2x_1 + 3x_2 + x_3 \le b_i\} \ge 0.85,$$
$$\Pr\{x_1 + x_2 + 2x_3 \le b_i\} \ge 0.90,$$
$$x_1, x_2, x_3 \ge 0\}.$$

Suppose that $b_i, (i = 1,2,3)$ are normally distributed random parameters with the following means and variances:

Table 1: The means and variances of (b_i)

Random variable	b_1	b_2	b_3
Mean	*6*	*10*	*7*
Variance	*9*	*25*	*16*

From standard normal tables, we have:

$K_{\alpha_1} = K_{0.92} \cong 1.4,\ K_{\alpha_2} = K_{0.85} \cong 1.02,\ K_{\alpha_3} = K_{0.90} \cong 1.3$

For the first constraint, the deterministic constraint is given by:

$$x_1 + x_2 + x_3 \le M_1 = E\{b_1\} + K_{\alpha_1}\sqrt{Var\{b_1\}}$$
$$= 6 + 1.4\sqrt{9}$$

For the second constraint:

$$2x_1 + 3x_2 + x_3 \le M_2 = E\{b_2\} + K_{\alpha_2}\sqrt{Var\{b_2\}}$$
$$= 10 + 1.02\sqrt{25}$$

For the third constraint:

$$x_1 + x_2 + 2x_3 \le M_3 = E\{b_3\} + K_{\alpha_3}\sqrt{Var\{b_3\}}$$
$$= 7 + 1.3\sqrt{16}$$

Therefore, a multi-level fractional programming problem with stochastic parameters in constraint can be understood as the corresponding (MLFPP) written as:

[FLDM]

$$\underset{x_1}{Max}\ F_1(x) = \frac{6x_1 + 3x_2 + 2x_3}{x_1 + x_2 + x_3 + 4},$$

Where x_2, x_3 solve

[SLDM]

$$\underset{x_2}{Max}\ F_2(x) = \frac{4x_1 + 5x_2 + 2x_3}{x_1 + x_2 + x_3 + 5},$$

Where x_3 solves

[TLDM]

$$\underset{x_3}{Max}\ F_3(x) = \frac{2x_1 + 3x_2 + 5x_3}{x_1 + x_2 + x_3 + 3},$$

Subject to

$$x_1 + x_2 + x_3 \le 10,$$
$$2x_1 + 3x_2 + x_3 \le 15,$$
$$x_1 + x_2 + 2x_3 \le 12,$$
$$x_1, x_2, x_3 \ge 0.$$

Now, we can use the 1^{st} order Taylor polynomial series to convert the (MLFPP) into (MLLPP) for the FLDM, SLDM and TLDM then $F_1^*(3,2,1)$, $F_2^*(3,2,2)$ and $F_3^*(2,2,1)$ as follows:

First, the FLDMsolves his/her problem as follows:

$$F_1(x) \cong \hat{F}_1(x) = F_1(3,2,1) + \begin{bmatrix} (x_1 - 3)\dfrac{\partial F_1(3,2,1)}{\partial x_1} \\ +(x_2 - 2)\dfrac{\partial F_1(3,2,1)}{\partial x_2} \\ +(x_3 - 1)\dfrac{\partial F_1(3,2,1)}{\partial x_3} \end{bmatrix}$$

$$F_1(x) \cong \hat{F}_1(x) = 0.34x_1 + 0.04x_2 - 0.06x_3 + 1.7$$

Whose solution is $(x_1^F, x_2^F, x_3^F) = (7.5, 0, 0)$

When putting this value in the objective function we have $F_1^* = 4.25$

Secondly, the SLDM defines his/her problem in view of the FLDM as follows:

$$F_2(x) \cong \hat{F}_2(x) = F_2(3,2,2) + \begin{bmatrix} (x_1-3)\dfrac{\partial F_2(3,2,2)}{\partial x_1} \\ +(x_2-2)\dfrac{\partial F_2(3,2,2)}{\partial x_2} \\ +(x_3-2)\dfrac{\partial F_2(3,2,2)}{\partial x_3} \end{bmatrix}$$

$$F_2(x) \cong \hat{F}_2(x) = 0.15x_1 + 0.24x_2 - 0.01x_3 + 1.3$$

Whose solution for the SLDMdoes the same action like the FLDM $(x_1^S, x_2^S, x_3^S) = (7.5, 5, 0)$

When putting this value in the objective function we have $F_2^* = 3.625$

Third, the TLDMdefines his/her problem in view of the SLDM as follows:

$$F_3(x) \cong \hat{F}_3(x) = F_3(2,2,1) + \begin{bmatrix} (x_1-2)\dfrac{\partial F_3(2,2,1)}{\partial x_1} \\ +(x_2-2)\dfrac{\partial F_3(2,2,1)}{\partial x_2} \\ +(x_3-1)\dfrac{\partial F_3(2,2,1)}{\partial x_3} \end{bmatrix}$$

$$F_3(x) \cong \hat{F}_3(x) = 0.02x_1 + 0.14x_2 + 0.4x_3 + 1.2$$

Whose solution for the TLDM does the same action like the SLDM $(x_1^T, x_2^T, x_3^T) = (7.5, 5, 6)$

When putting this value in the objective function we have $F_3^* = 4.45$

VI. Summary and Concluding Remarks

This paper proposed an approach for solving a multi-level fractional programming problem (MLFPP) involving stochastic parameters coefficient in constraints. Such problems are called optimization under chance constraints. At the first phase of the solution approach, we converted the probabilistic nature of this problem into a multi-level fractional programming problem (MLFPP).At the second phase, we used the 1^{st} order Taylor series polynomial series to convert (MLFPP) into a multi-level linear programming problem (MLFPP) for generating a compromise solution for this problem. Finally, a numerical example was provided to demonstrate the correctness of the proposed solution.

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