Topological Control and Stability of Dynamic Fractional Order Systems with Generalized Memory

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Abstract – In the context of scientific direction-physic open system are considered the topological control and stability chaotic fractional nonlinear system under the criterion Poincare return time.

The results were obtained by non-analytical methods. The analysis and synthesis made by use the observed fractional-order Genesio-Tesi system. Suggested that generalized compact metric space of Poincare.

Keywords – Fractional-Order Genesio-Tesi System, Topological Control And Stability, Poincare Space, Generalized Memory.

I. INTRODUCTION

In the context of scientific direction-physic open systems [1] are considered the analysis and synthesis of fractional physical systems under the criterion Poincare return time [2,3].

We know that the equations in fractional derivatives describe the evolution of physical system losses and fractional rate of derivatives indicates the share of the remaining states of the system for the time evolution. Such systems are called systems for the time evolution. Such systems are called systems with “residual” memory occupying an intermediate position memory on the one band and Markov system on the other.

In the context of use of fractional dynamics in research problems of Physical systems there is a problem of returns of Poincare. Times of returns represent the main indicators and characteristics showing as certain conditions of dynamic system repeat on time.

In this paper in the paradigm formed the ideological basis of which is the realization of topological problems of topological control and stability of nonlinear chaotic fractional-order system, in the context of the use of such criteria as Poincare return time, Lyapunov function, generalized memory function [4].

Below the main notation and preliminaries.

II. NOTATION AND PRELIMINARIES

2.1. We consider the fractional-order Genesio-Tesi system [5]:

$$\frac{d^\alpha x_i}{dt^\alpha} = x_i, \quad \frac{d^\beta x_i}{dt^\beta} = x_i, \quad \frac{d^q x_3}{dt^q} = -\beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3 + \beta_4 x_4^2,$$

where: $0 < q \leq 1$, $\beta_1, \beta_2, \beta_3$ and $\beta_4$ are system parameters.

In the following simulations in this section, we only vary derivative order $q$ and the system parameter $\beta$, the other system parameters are fixed.

Implementation of the system is performed by applying the operator Riemann-Liouville [5]:

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} \frac{f(\tau)}{(t-\tau)^{n-\alpha}} d\tau,$$

for $n-1 \leq \alpha < n$, where $\Gamma(\cdot)$ is the Gamma function.

2.2. Consider the following general structure of the fractional-order nonlinear system under control:

$$D^\alpha x(t) = f(x(t)) + Bu(t),$$

where $u(t) = \left[u_1(t) u_2(t) ... u_n(t)\right]^T$ is $m$-dimensional input vector that will be used and following control structure will be considered for state feedback:

$$u(t) = u_\nu(t) + u_\alpha(t),$$

where $u_\nu(t)$ is equivalent control and $u_\alpha(t)$ is the switching control of the system (2).

Definition 1 [6]. The observed $x_{\alpha} = \left(x_n \right)_{n=0}^w$ is called deterministically if the following conditions:

- there is graded dynamic system $f$ point $y_n$ and the function $\varphi(f^n(y_0)) = x_n$, for all $n = 1, 2, ...$;
- the distance $dist(f^n x, f^n x') \leq const \cdot e^{ik} \times dist(x, x')$, i.e. maximum Lyapunov charts allow must be limited;
- function $\varphi$ Lipschitz – continuous, i.e. $|\varphi(x) - \varphi(x')| \leq L|x - x'|$ function known as the constant of Lipschitz.

Thus, the study observed may provide an answer to the fundamental question of the limbs and determinacy process.

III. MAIN RESULT

3.1. Topology fractional-order space

Hausdorff-Bezikovich's dimension [7] is metric concept, but there is its fundamental connection with topological dimension $dim E$ which was established by L.S.Pontrjagin and L.S.Shnirelman, having entered concept of a metric order [7].
Definition 2. The number is called as a metric order of a compact \( A \)
\[
k = \lim(\ln N_{\varepsilon}(\varepsilon)/\ln \varepsilon),
\]
where \( \varepsilon \) - the sphere of radius \( \varepsilon \); \( N(\varepsilon) \) - number of spheres in a final sub covering of a set.
The lower bound of metric orders for all metrics of a compact \( A \) (called by metric dimension) is equal his Lebesgue to dimension.
However it appeared that the metric order entered in [7], coincides with the lower side the fractal dimension of Hausdorff-Bezikovich defined in the terms “box-counting”.

Takes place

Theorem 1 [7]. For any compact metric space \( X \).
\[
dim X = \inf \left\{ \lim_{\varepsilon \to 0} -\log d \mid d \text{ is a metric on } X \right\},
\]
where
\[
N_{\varepsilon}(X) = \min \left\{ |U| : U \text{ is a finite open covering of } X \text{ with mesh } \leq \varepsilon \right\}
\]
From here \( (X, d_{X}) \) - compact fractal metric space with dimension \( d_{X} \).

Here it is important to note that at the description of properties of systems with fractional structure it is impossible to use representation of Euclidean geometry.
There is a need of the analysis of these processes for terms of geometry of fractional dimension.

Remark. In [8] presented results of communication of a fractional integrodifferentiation (in Rimana-Liouville or Grunvalda-Letnikov’s terms) with Koch’s curves.

It is noted that biunique communication between fractals and fractional operators does not exist: fractals can be generated and described without use of fractional operations, and defined the fractional operator not necessarily generates defined (unambiguously with it connected) fractal process or fractal variety.

However use of fractional operations allows generating other fractal process (variety) which fractional dimension is connected with an indicator of a fractional integro differentiation a linear ratio on the basis of the set fractal process (variety).

In [8] fractional integrals of Rimann-Liouville are understood as integrals on space of fractional dimension.
Thus the indicator of integration is connected with dimension of space an unambiguous ratio.

In this regard consideration of dimension of chaotic systems of a fractional order causes interest. So, in [9] was noted that dimension of such systems can be defined by the sum of fractional exponents \( \Sigma \), and \( \Sigma < 3 \) is the most effective.

Let the chaotic fractional system of Lorentz take place [9]:
\[
\begin{align*}
d^{\alpha}x &= \sigma(y-x), \\
d^{\beta}y &= \rho x - y - xz, \\
d^{\xi}z &= xy - bz
\end{align*}
\]
here \( \sigma = 10, \rho = 28, b = 8/3; \ 0 < \alpha, \beta, \gamma \leq 1, \ r \geq 1. \)

Then fractional dimension of system of the equations (6) will have an appearance [9]:
\[
\alpha + \beta + \gamma = \Sigma.
\]
So, for example, for Lorentz’s system with fractional exponents \( \alpha = \beta = \gamma = 0.99 \), effective dimension \( \Sigma = 2.97 \).

This, in the context of fractional dynamics let \( \tilde{X} \) - any set of nonlinear physical systems, \( A^{\varepsilon} \) - a subset of a set \( \tilde{X} \) of systems of a fractional order with memory \( A^{\varepsilon} \subset \tilde{X} \). Then a triad \( (\tilde{X}, A^{\varepsilon}, \Sigma) \) compact fractional metric space with dimension \( \Sigma \).

Let’s designate \( W \in (X, d_{X}) \). On the basis [10] and remarks \((\tilde{X}, A^{\varepsilon}, \Sigma) \subset W \).

Let’s consider transformation \( W \) at an angle of communications of average time of return of Poincare \( \langle \tau \rangle \) with \( d_{j} \) and “residual” general memory (GM).

Here \( g : (\tau) \Rightarrow d_{j} l: d_{j} \Rightarrow GM \). \( \chi : (\tau) \Rightarrow (g, l) \).

From here \( U \in (X, \langle \tau \rangle) \) - the generalized compact metric space of Poincare with dimension \( \langle \tau \rangle \).

3.2. Generalized systems with memory

Let \( (\tilde{X}, A^{\varepsilon}, \Sigma) \subset W \); \( Z \) the set of all integers:
\[
R_{\infty} = [0, \infty), \quad R_{0} = (\infty, 0), \quad Z_{\infty} = [1, 2, \ldots], \quad Z_{0} = \{0, -1, -2, \ldots\}.
\]

Definition 3. Let \( (\tilde{X}, A^{\varepsilon}, \Sigma) \subset W \) be GM, \( GM \subset R \times Z \) is called a compact generalized memory of [4]:
\[
GM = Q_{\infty} \bigcup Q_{0},
\]
where
\[
Q_{\infty} = \bigcup_{j=0}^{\infty} \left\{ \left[ t_{j}, t_{j+1} \right] \right\},
\]
and
\[
Q_{0} = \bigcup_{j=1}^{\infty} \left\{ \left[ s_{j-1}, s_{j} \right] \right\} - k + 1
\]
for same finite of observed:
\( s_{1} \leq \ldots \leq s_{k} \leq s_{0} = 0 = t_{0} \leq t_{1} \leq \ldots \leq t_{j} \).
piece-wise function $S(t_i)$, (fig.1), $S_i : R \rightarrow R$, $i = 1,2,\ldots,n$, real piece-wise constant function, \( L(x(t))GM \) - Lyapunov stability function for solution fractional-order system matches $GM$. Let \( U \in (X_\tau^\tau) \) is tensioned the mathematical model \( D^\tau x(t) \), i.e. \[
D^\tau x(t) \rightarrow U .
\] (10)

Remark. Important in the theory of nonlinear physical fractional-order systems is the use the dynamics caused by their observed. Continuous flow in phase space determines the behavior of the system can be studied using a discrete display the induced flow at the section of Poincare. It is a relativistic terms in the rights continuous flow and its discrete in time delay.

This feature makes it easy solution to complex problems.

IV. ALGORITHM

\textbf{Step 1.} Let \( \hat{x} = \{x_n\}_{n=0}^N \) is reference observed with \( d_i^{ref} \ = \ 1.3120 \), \( \{\tau\}^{ref} \ = \ 4.080 \).

\textbf{Step 2.} Simulation of the system (2) according to the algorithm [11].

\textbf{Definition 4.} The system topologically controllable if and only if coincides with \( \hat{x} \) on the basis of the criterion metrics “proximity” Hausdorff.

\textbf{Step 3.} The resulting observable \( q \cdot \hat{y} = \{y_n\}_{n=0}^N \) perturb of piece-wise ordered: \( S = \{s_n\}_{n=0}^N \), \( \bar{Y} = q \cdot \hat{y} + S \), \( \bar{Y} = \{\bar{y}_n\}_{n=0}^N \), \( \bar{Y} \in U \).

\textbf{Step 4.} Determine the fractal dimension \( d_f \) and Poincare return time \( \{\tau\} \) [2]: \( d_f \ = \ 1.4581 \); \( \{\tau\} = -d_f \ln \varepsilon \), \( \varepsilon > 0 \), \( \{\tau\} = 4.6054 \).

\textbf{Theorem 2} [12]. Let \( E \) and \( F \) is compact subset \( R^n \), \( \varepsilon > 0 \). Hausdorff distance \( H(E,F) \) satisfies the relation.

\( H(E,F) \leq \hat{\varepsilon} \Leftrightarrow E \subset F + \hat{\varepsilon} \) and \( F \subset E + \hat{\varepsilon} \), where \( \hat{\varepsilon} > 0 \) the allowable threshold.

\textbf{Step 5.} Produce operate of “comparison on the standard” use Hausdorff fractal metric of “proximity”.

\( H_0(d_f,d^{ref}_f) = \emptyset \).

\textbf{Step 6.} ILC and criterion $AI$ (area of interest) [13, 14] to implement the tasks of “comparison on the standard” \( AI(y)^{def} = \sup_{\varepsilon \in \mathcal{V}} I \) i.e. the most significant information from semantic point of view the \( \sup_{\varepsilon \in \mathcal{V}} \) and minimum volume data \( \inf_{\varepsilon \in \mathcal{V}} I \) (fig. 1).

Fractional-order iterative learning control scheme is given as [13]:

\[
U_{i+1}(t) = F(U_i(t), e_i(t)),
\]

\[
e_i(t) = Y_d(t) - Y_i(t).
\]

Fig. 1. The basic scheme of iterative learning control with \( Y_i(t) \) being the trajectory, \( U_i(t) \) and \( Y_i(t) \) the input signal.

Hence the memory loss will be determined by the difference between the global and the local area supported by that is reversibility and irreversibility of the process.

\( lm_1 = d_f - d(AI_1) \), \( H_1(d(AI_1), d^{ref}_f) = \emptyset \); \( lm_2 = d_f - d(AI_2) \), \( H_2(d(AI_2), d^{ref}_f) = \emptyset \); \( lm_3 = d_f - d(AI_3) \), \( H_3(d(AI_3), d^{ref}_f) \neq \emptyset \),

\( \text{where } lm \) - loss memory.

\textbf{Step 7.} Topological control attainable, if it satisfies:

\( H(d_f,d^{ref}_f) \leq \hat{\varepsilon} \Leftrightarrow d^{ref}_f \subset d_f + \hat{\varepsilon} \) and \( d_f \subset d^{ref}_f + \hat{\varepsilon} \),

where \( \hat{\varepsilon} = 0.110 \); \( \Delta(\tau) \ = \ 0.098 \).

\textbf{Step 8.} Displayed the \( \bar{Y} \) on the two-dimensional square matrices \([N, N]\) and of formula [15]:

\( R_{i,j}^{E} = \theta(\varepsilon_i - \|x_i - x_j\|) \), \( i, j = 1,\ldots,N \), \( i \neq j \), \( x \in U \),

where \( N \) - number of considered (examined) condition \( x_i ; \varepsilon \) - size of a neighborhood of a point \( x \) at the moment \( i; || \cdot || \) - norm; \( \theta(\cdot) \) - function of Heaviside.

Obtain a diagram Poincare of fractional order \( q \cdot D(\bar{Y}) \in U \) (fig.2).
Thus the implementation of tasks received a generalized memory structure (fig.2).

Fig.2. a) fractional Poincare diagram; b) spectrum of Poincare return time \( \langle \tau \rangle \); c) topology of the case of regular behavior patterns.

Fig.3. Generalized memory: \([-452, 0]\) – loss memory, \([0, 120]\) – memory.

\[ \frac{dx}{dt} = \hat{\Omega}, \quad \hat{\Omega} = \{\hat{\omega}_n\}_{n=0}^N, \quad \hat{\Omega} \subset \theta \psi. \]

\( \langle \tau \rangle \) - if trajectory will pass through the point 0, i.e. \( \dot{V}(x) \leq 0 \), the system is stable with \( d_{\langle \tau \rangle} \) and matches \( GM \).

\( \langle \tau \rangle \) - if trajectory will pass below the point 0, the system is asymptotically stable with \( d_{\langle \tau \rangle} \).

**Proof**

**Step 1.** Let the Lyapunov function is given in the quadratics form as:

\[ V(x) = q \hat{\Omega}^2. \]

**Step 2.** We calculate the total derivative of the function:

\[ \frac{dV}{dt} = \frac{\partial V}{\partial x} \frac{dx}{dt} = 2q \hat{\Omega} \hat{\psi} \leq 0. \]

Fig.4 shows the trajectory of function Lyapunov, satisfying paragraph (i) of the Proposition.

**VI. CONCLUSION**

Thus, at excitement by parameters \( S \) and \( AI \) on Genesio-Tesi system, control to receive a regular Lyapunov component at the exit at the corresponding losses of memory \( lm \) and satisfaction of the parameters \( d_{\text{ref}} \) and \( \langle \tau \rangle \) reference \( d_{\text{ref}}^{\text{ref}} \) and \( \langle \tau \rangle^{\text{ref}} \).

On the basis of the conducted researches the conclusion was drawn that resonant excitement directly don't regulate behavior of the operated system, and only form the mechanism of its self-organization, i.e. promotes the organization of new structures.

First suggested that generalized compact metric space of Poincare. Defined topological control and stability of fractional-order system with generalized memory. An implementation task is accompanied by a visual display in terms of nonlinear recurrence analysis.

Algorithms of topological control and stability are realized in the MATLAB.

**REFERENCES**


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