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Harmonic Lamb Waves in Heterogeneous Anisotropic Plates

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***Abstract* – Theoretical methods for analyses of Lamb waves propagating in multilayered elastic anisotropic plates are reviewed. The global and transfer matrix methods used for analyzing Lamb waves in laminated plates along with the associated numerical problems are discussed.**

***Keywords* – Lamb Wave, Multi-Layered Plate, Anisotropy, Dispersion, Stroh Formalism, Cauchy Formalism, Transfer Matrix, Global Matrix.**

I. Lamb Waves in a Homogeneous Isotropic Plate

The first works [1, 2] on waves propagating in an infinite isotropic homogeneous plate with the traction-free boundary surfaces were done at assumption that the wavelength is much longer than the plate thickness.

The complete theory of harmonic Lamb waves free from the long wavelength limit assumption was presented in [3]. The starting point of the Lamb theory is considering the equation of motion in the form

$$c_P \nabla \operatorname{div} \mathbf{u} - c_S \operatorname{rot} \operatorname{rot} \mathbf{u} = \frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{1}$$

where $\mathbf{u}$ is the displacement field, c_P and c_S are velocities of the longitudinal and transverse bulk waves respectively:

$$c_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_S = \sqrt{\frac{\mu}{\rho}}. \tag{2}$$

In (2) λ and μ are Lamé constants and ρ is the material density. Then, the displacement field was represented in terms of scalar (Φ) and vector ($\boldsymbol{\Psi}$) potentials

$$\mathbf{u} = \nabla \Phi + \operatorname{rot} \boldsymbol{\Psi}. \tag{3}$$

The potentials were assumed to be harmonic in time

$$\Phi(\mathbf{x},t) = \Phi'(\mathbf{x}) e^{i\omega t}, \quad \boldsymbol{\Psi}(\mathbf{x},t) = \boldsymbol{\Psi}'(\mathbf{x}) e^{i\omega t}. \tag{4}$$

Substituting representation (4) into Eq. (1) yields two independent Helmholtz equations

$$\left(\Delta + \frac{\omega^2}{c_P^2}\right)\Phi' = 0, \quad \left(\Delta + \frac{\omega^2}{c_S^2}\right)\boldsymbol{\Psi}' = 0. \tag{5}$$

To define the spatial periodicity and to simplify the analysis, the splitting spatial argument is needed

$$\mathbf{x} = (\mathbf{x}\cdot\mathbf{n})\mathbf{n} + (\mathbf{x}\cdot\boldsymbol{\nu})\boldsymbol{\nu} + (\mathbf{x}\cdot\mathbf{w})\mathbf{w}, \tag{6}$$

where $\mathbf{n}$ is the unit wave vector, $\boldsymbol{\nu}$ is the unit normal to the median plane of the plate, and $\mathbf{w} = \mathbf{n} \times \boldsymbol{\nu}$.

Remark 1. For the considered waves it was further assumed that the displacement field does not depend upon argument $\mathbf{x}\cdot\mathbf{w}$. That allowed Lamb [3] to consider scalar potentials Ψ and Ψ' in (4) instead of vector ones (actually, Lamb considered the vector potential composed of one non-vanishing component $(\boldsymbol{\Psi}\cdot\mathbf{w})\mathbf{w}$).

The further assumption relates to the periodicity of the potentials in the direction of propagation

$$\Phi'(\mathbf{x}) = \varphi(x'')e^{x'}, \quad \Psi'(\mathbf{x}) = \psi(x'')e^{x'}, \tag{7}$$

where the dimensionless complex coordinates x' and x'' are

$$x' = ir\,\mathbf{x}\cdot\mathbf{n}, \quad x'' = ir\,\mathbf{x}\cdot\boldsymbol{\nu}. \tag{8}$$

In (8) $i = \sqrt{-1}$ and r is the wave number related to the wavelength l by

$$r = \frac{2\pi}{l}. \tag{9}$$

Substituting representations (7) into Eq. (5) results in the decoupled ordinary differential equations

$$\frac{d^2\varphi}{dx''^2} + \left(1 - \frac{c^2}{c_P^2}\right)\varphi = 0, \quad \frac{d^2\psi}{dx''^2} + \left(1 - \frac{c^2}{c_S^2}\right)\psi = 0, \tag{10}$$

where the phase speed c relates to the frequency and the wave number by the following relation

$$c = \frac{\omega}{r} \tag{11}$$

The general solution of Eqs. (10) can be written the form

$$\begin{aligned} \varphi(x'') &= C_1 \sinh(\gamma_1 x'') + C_2 \cosh(\gamma_1 x''), \\ \psi(x'') &= C_3 \sinh(\gamma_2 x'') + C_4 \cosh(\gamma_2 x'') \end{aligned} \tag{12}$$

where

$$\gamma_1 = \left(1 - \frac{c^2}{c_P^2}\right)^{1/2}, \quad \gamma_2 = \left(1 - \frac{c^2}{c_S^2}\right)^{1/2} \tag{13}$$

The unknown coefficients in (12) are defined (up to a multiplier) from the following boundary conditions on the free surfaces

$$\begin{aligned} \mathbf{t}_\nu &\equiv \left(\lambda \operatorname{tr}(\nabla\mathbf{u})\mathbf{I} + \mu\left(\nabla\mathbf{u} + \nabla\mathbf{u}^t\right)\right)\cdot\boldsymbol{\nu} = 0 \\ \mathbf{x}\cdot\boldsymbol{\nu} &= \pm h \end{aligned}, \tag{14}$$

where $2h$ is the depth of the plate. Substitution representation (3) into boundary conditions (14) yields boundary conditions written in terms of potentials φ'' and ψ''

$$\left(\begin{array}{l}\lambda\Delta\Phi'\mathbf{I}+\\ 2\mu\left(\nabla\nabla\Phi'+\frac{1}{2}\left(\nabla\,\mathrm{rot}\,\mathbf{\Psi}'+\left(\nabla\,\mathrm{rot}\,\mathbf{\Psi}'\right)^{t}\right)\right)\end{array}\right)\cdot\mathbf{v}=0 \qquad .(15)$$

$$\mathbf{x}\cdot\mathbf{v}=\pm h$$

Substituting solutions (12) into Eq. (15) in view of Remark 1, yields the desired dispersion equation, found in [1, 3]:

$$\frac{\tanh(\gamma_2 rh)}{\tanh(\gamma_1 rh)}-\left(\frac{4\gamma_1\gamma_2}{(1+\gamma_2^2)^2}\right)^{\pm 1}=0. \qquad (16)$$

The sign "+" refers to symmetrical, and "−" to anti-symmetrical modes. In view of expressions (11) and (13) the obtained dispersion equation implicitly determines the phase velocity c as a function of frequency. Equations for velocities related to the long wave and short wave limits were found in [3].

Taking the short wave limit at $rh\to\infty$ in (16) yields

$$1=\left(\frac{4\gamma_1\gamma_2}{(1+\gamma_2^2)^2}\right). \qquad (17)$$

The latter expression coincides with the secular equation for the speed of Rayleigh waves derived in [4], and hence the first limiting speed coincides with the speed of Rayleigh wave

$$c_{1,\lim}=c_R. \qquad (18)$$

Analysis of Eq. (16) at $rh\to 0$ (the long wave limit) yields the following equation [5]

$$\frac{\gamma_2}{\gamma_1}=\left(\frac{4\gamma_1\gamma_2}{(1+\gamma_2^2)^2}\right), \qquad (19)$$

from where the values for two limiting velocities can be obtained

$$c_{2,\lim}=2c_S\sqrt{1-\frac{c_S^2}{c_P^2}}. \qquad (20)$$

$$c_{3,\lim}=c_S. \qquad (21)$$

Expression (20) coincides with the limiting velocity value found in [1, 2] and differs from the anticipated value for the long wave limiting velocity of the sound waves in rods.

For the anti-symmetric fundamental mode, Eq. (16) was analytically solved in [6] using the perturbation technique. Earlier studies of the dispersion curves for Lamb waves in a layer contacting with a halfspace were mainly associated with the geophysical applications [7 - 10]. The analysis of the dispersion curves at different Poisson's ratios (including negative values) was done in [11 - 13]. The points of intersection of the dispersion curves were studied in [14].

Consider now the notion of the group velocity introduced by Stokes [15] for description of the wave package propagation in hydrodynamics and later extrapolated to acoustic waves in the theory of elasticity; see [16 - 18]. Formally, the group velocity can be defined by

$$c^{group}=\frac{d\omega}{dr}. \qquad (22)$$

Numerical studies [19 - 25] of the group velocity dispersion, mainly at Poisson's condition $\lambda=\mu$, confirmed Rayleigh's anticipation [16, 17] that the negative values of the group velocity can appear at the very small wave numbers. Later on, numerical computations [25] revealed existence of a broader range of negative group velocities at Poisson's ratios belonging to the interval $0.31<\nu<0.45$.

Several additional equations for computing dispersion of the group velocity can be constructed from Eq. (16). For example, substituting the phase speed defined by Eq. (11) in Eqs. (13), denoting the left-hand side of Eq. (16) by $F^{\pm}(r,\omega)$, and assuming that ω is a function of r, the derivative of Eq.(16) with respect to r takes the form

$$\frac{\partial F^{\pm}(r,\omega)}{\partial r}+\frac{\partial F^{\pm}(r,\omega)}{\partial\omega}\frac{d\omega}{dr}=0, \qquad (23)$$

from where in view of (22), the secular equation for the group speed takes the form

$$c^{group}=-\frac{\dfrac{\partial F^{\pm}(r,\omega)}{\partial r}}{\dfrac{\partial F^{\pm}(r,\omega)}{\partial\omega}}. \qquad (24)$$

In (24) r is considered as a function of ω. Theoretical studies of the phase, group, and ray velocities were done in [26].

II. Homogeneous Anisotropic Plate, Three-Dimensional Formalism

In the first works on Lamb waves propagating in anisotropic plates a three dimensional formalism was used. Initially, that formalism was developed in [27] for analysis of Rayleigh waves in an anisotropic halfspace; later on this method was applied to halfspaces with different groups of elastic symmetry [28 - 32]. With necessary modification, the approach [27] was used for analysing Lamb waves in anisotropic plates [33 – 43].

All these publications, except [33] where a more complicated case of a cylindrically anisotropic plate was considered, actually exploited the following representation for the displacement field

$$\mathbf{u}(\mathbf{x},t)=\left(\sum_{k=1}^{6}C_k\underbrace{\mathbf{m}_k e^{\gamma_k x''}}_{\mathbf{u}_k(x'')}\right)e^{i(r\mathbf{n}\cdot\mathbf{x}-\omega t)} \qquad (25)$$

where C_k are arbitrary complex coefficients determined up to a multiplier by satisfying conditions at the plate boundaries; $\mathbf{u}_k$ is the displacement field of the k-th partial wave; $\mathbf{m}_k$ is a vectorial, generally, complex amplitude, determined by the Christoffel equation (this equation will be introduced later on); γ_k is a root of the Christoffel equation. Note, that according to Eq. (8), the coordinate x'' is imaginary. Six partial waves in (25) correspond to six (not necessary aliquant) roots of the Christoffel equation.

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Substituting representation (25) into equation of motion

$$\mathbf{A}(\partial_x, \partial_t)\mathbf{u} \equiv \operatorname{div}_x \mathbf{C} \cdot\cdot \nabla_x \mathbf{u} - \rho \ddot{\mathbf{u}} = 0, \qquad (26)$$

where $\mathbf{C}$ is the fourth order elasticity tensor assumed to be positive definite, yields the Chrisoffel equation

$$\left[(\gamma_k \mathbf{v} + \mathbf{n}) \cdot \mathbf{C} \cdot (\mathbf{n} + \gamma_k \mathbf{v}) - \rho c^2 \mathbf{I} \right] \cdot \mathbf{m}_k = 0, \qquad (27)$$

where $\mathbf{I}$ is the unit diagonal matrix. Equation (27) admits the equivalent form

$$\det\left[(\gamma_k \mathbf{v} + \mathbf{n}) \cdot \mathbf{C} \cdot (\mathbf{n} + \gamma_k \mathbf{v}) - \rho c^2 \mathbf{I} \right] = 0. \qquad (28)$$

The left-hand side of Eq. (28) represents a polynomial of degree six with respect to the Christoffel parameter γ_k. Equations (27), (28) show that roots γ_k and the corresponding eigenvectors $\mathbf{m}_k$ can be considered as functions of the phase speed c.

Remark 2. a) For Rayleigh waves the roots γ_k in representation (25) should be complex with $\operatorname{Im}(\gamma_k) < 0$, this ensures attenuation of Rayleigh wave in the "lower" half-space $(\mathbf{v} \cdot \mathbf{x}) < 0$. The condition $\operatorname{Im}(\gamma_k) < 0$ confines the admissible speed interval and decreases the number of summation terms in (25). If $\operatorname{Re}(\gamma_k) = 0$ for all partial waves composing Rayleigh wave, then such a wave is called the genuine Rayleigh wave; if $\operatorname{Re}(\gamma_k) \neq 0$ for some k, then such a wave is called the generalized Rayleigh wave [28]. For Lamb waves the cases $\operatorname{Re}(\gamma_k) = 0$ and $\operatorname{Re}(\gamma_k) \neq 0$ are usually not distinguished.

b) Within the discussed formalism the case of appearing multiple roots γ_k and the coincident kernel eigenvectors $\mathbf{m}_k$ was considered in [44] with application to Rayleigh waves and in [45] for obtaining the dispersion equation of subsonic Lamb waves.

c) For anisotropic plates the limiting velocities (20), (21) were computed in [46, 47].

To obtain the dispersion equation, consider the traction-free boundary conditions

$$\mathbf{t}_\nu \Big|_{\mathbf{x}\cdot\mathbf{v} = \pm h} \equiv \pm \mathbf{v} \cdot \mathbf{C} \cdot\cdot \nabla_{\mathbf{x}} \mathbf{u} \Big|_{\mathbf{x}\cdot\mathbf{v} = \pm h} = 0, \qquad (29)$$

where $2h$ denotes the overall thickness of the plate. Substituting representation (25) into the boundary conditions (29), yields the dispersion equation in the form

$$\sum_{k=1}^{6} C_k \left(\gamma_k \mathbf{v} \cdot \mathbf{C} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{C} \cdot \mathbf{n} \right) \cdot \mathbf{m}_k e^{\pm i r \gamma_k h} = 0. \qquad (30)$$

Finally, Eq. (30) can be rewritten in a form better suited for numerical computations

$$\det(\mathbf{M}) = 0. \qquad (31)$$

where $\mathbf{M}$ is a 6x6-matrix of the 3D-formalism. Dispersion equation (31) defines the phase speed as a function of the wave number, or in view of (11), as a function of the circular frequency.

III. Homogeneous Anisotropic Plate, Stroh Six-Dimensional Formalism

Initially Stroh formalism [48] was applied to analysis of Rayleigh waves propagating on a free boundary of an anisotropic halfspace [49 - 55]. The case of the non-semisimple degeneracy of the fundamental matrix was considered in [53] (that case is associated with appearing multiple roots and the coincident kernel eigenvectors in the Christoffel equation). In [56, 57] the Stroh formalism was applied to description of Lamb waves propagating in the homogeneous anisotropic plates.

Following [49] the displacement field for Rayleigh or Lamb wave is searched in the form

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{f}\left(x''\right) e^{i(r\,\mathbf{n}\cdot\mathbf{x} - \omega t)} \qquad (32)$$

with the unknown amplitude $\mathbf{f}$ regarded as a function of the imaginary coordinate x'' defined by (8). Substituting representation (32) into equation of motion (26), yields

$$\left(\mathbf{A}_1 \partial_{x''}^2 + \left(\mathbf{A}_2 + \mathbf{A}_2^t \right) \partial_{x''} + \mathbf{A}_3 \right) \cdot \mathbf{f}(x'') = 0, \qquad (33)$$

where

$$\mathbf{A}_1 = \mathbf{v} \cdot \mathbf{C} \cdot \mathbf{v}, \quad \mathbf{A}_2 = \mathbf{v} \cdot \mathbf{C} \cdot \mathbf{n}, \quad \mathbf{A}_3 = \mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n} - \rho c^2 \mathbf{I}. \qquad (34)$$

Matrix $\mathbf{A}_1$ is non-degenerate due to the assumption of positive definiteness of the elasticity tensor.

Remark 3. For an isotropic material, matrices (34) take the following form

$$\mathbf{A}_1 = \begin{pmatrix} \lambda + 2\mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 0 & \lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A}_3 = \begin{pmatrix} \mu - \rho c^2 & 0 & 0 \\ 0 & \lambda + 2\mu - \rho c^2 & 0 \\ 0 & 0 & \mu - \rho c^2 \end{pmatrix} \qquad (35)$$

Up to the harmonic multiplier $e^{i(r\,\mathbf{n}\cdot\mathbf{x} - \omega t)}$, the surface tractions on the planes parallel to the free boundary (or the median surface of a plate) can be written in terms of matrices (34)

$$\mathbf{t}_\nu(x'') \equiv \left(\mathbf{A}_1 \partial_{x''} + \mathbf{A}_2 \right) \cdot \mathbf{f}(x''). \qquad (36)$$

The main idea of Stroh formalism is in rewriting the equation of motion (26) in terms of the displacements and surface tractions. Multiplying both sides of (36) by matrix $\mathbf{A}_1^{-1}$ yields the following expression for the derivative $\partial_{x''} \mathbf{f}(x'')$

$$\partial_{x''} \mathbf{f}(x'') = \mathbf{A}_1^{-1} \cdot \mathbf{t}_\nu(x'') - \mathbf{A}_1^{-1} \cdot \mathbf{A}_2 \cdot \mathbf{f}(x''). \qquad (37)$$

Combining now Eqs. (33) - (37), produces the desired equation of motion written in terms of vectors $\mathbf{f}$ and $\mathbf{t}_\nu$

$$\partial_{x''} \begin{pmatrix} \mathbf{f} \\ \mathbf{t}_\nu \end{pmatrix} = \mathbf{N} \cdot \begin{pmatrix} \mathbf{f} \\ \mathbf{t}_\nu \end{pmatrix}, \qquad (38)$$

where $\mathbf{N}$ is the fundamental matrix [58]:

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$$\mathbf{N}=\begin{pmatrix} -\mathbf{A}_1^{-1}\cdot\mathbf{A}_2 & \mathbf{A}_1^{-1} \\ \mathbf{A}_2^t\cdot\mathbf{A}_1^{-1}\cdot\mathbf{A}_2-\mathbf{A}_3 & -\mathbf{A}_2^t\cdot\mathbf{A}_1^{-1} \end{pmatrix}. \qquad (39)$$

Now, the general solution of Eq. (38) can be represented in the form

$$\begin{pmatrix} \mathbf{f} \\ \mathbf{t}_\nu \end{pmatrix}=\exp(x''\mathbf{N})\cdot\vec{C}_6\,, \qquad (40)$$

where $\vec{C}_6$ is the six-dimensional generally complex vector of the unknown coefficients; that vector can be defined (up to a multiplier) by the boundary conditions. Substituting solution (40) into boundary conditions (29), yields

$$\begin{aligned} \begin{pmatrix} \mathbf{f}(+irh) \\ \mathbf{0} \end{pmatrix}&=\exp(+irh\mathbf{N})\cdot\vec{C}_6 \\ \begin{pmatrix} \mathbf{f}(-irh) \\ \mathbf{0} \end{pmatrix}&=\exp(-irh\mathbf{N})\cdot\vec{C}_6 \end{aligned}\,. \qquad (41)$$

Excluding vector $\vec{C}_6$ from Eq. $(41)_1$ and substituting the resultant expression in $(41)_2$ gives the following dispersion equation for an anisotropic plate

$$\det\left((\mathbf{0},\mathbf{I})\cdot\exp(-2irh\mathbf{N})\right)\cdot\begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix}=0\,. \qquad (42)$$

Despite the obvious simplicity of deriving Eq. (42), the relevant works on Lamb waves in anisotropic plates [56, 57] use a more complicated procedure.

IV. Multilayered Plate, Transfer Matrix Method

The transfer matrix method suggested for analyzing propagation of Lamb and Rayleigh waves multilayered *isotropic* media was introduced in [59, 60]. Up to now, the transfer matrix method was combined with the three-dimensional formalism. The main idea of the method is to exclude the unknown coefficients C_k in representation (25) expressing them in terms of the only partly known surface tractions and displacements on one of the outer surfaces of the plate; such a procedure is similar to (40), (41). If the plate contains several layers, the interface conditions can also be expressed in terms of these surface tractions and displacements via specially constructed transfer matrices. Finally, the boundary conditions on the other outer surface are expressed in terms of the surface tractions and displacements from the first outer surface.

Since its introduction, the transfer matrix method was applied to finding the dispersion relations for Lamb waves propagating in both isotropic [60-62] and monoclinic (or with higher symmetries) [63, 64] multilayered plates. Despite the abundance and simplicity, the original variant of the transfer matrix method revealed some numerical instability, especially when high frequencies and large depths of the layers were considered. To overcome this problem, the numerically stable algorithms were suggested [65-70]. Differing in details, these algorithms have the common principle idea of eliminating the terms containing large exponentials. Such a procedure developed in [65], is known as the δ-matrix method [66, 67]. The δ-matrix method can also be applied for analyzing dispersion of Lamb waves in a single layered plate; see [65].

In the sections below, the transfer matrix method will be coupled with the Cauchy six-dimensional formalism, and a numerically stable approach resembling [65-70] will be worked out.

V. Multilayered Plate, Global Matrix Method

The global matrix method introduced in [71], appeared more numerically stable than the original version of the transfer matrix method; see [62, 65] for discussions. In [72-77] applications of the global matrix method to analyses of Lamb wave dispersion are presented.

There are several variants of constructing the global matrix. For a traction-free plate containing n-homogeneous anisotropic layers, the matrix equation related to the global matrix method can be written in the following form:

$$\vec{0}_{6n}=\begin{pmatrix} (\mathbf{0},\,\mathbf{I})\cdot\mathbf{D}_1^+ & & & & \\ \mathbf{D}_1^- & -\mathbf{D}_2^+ & & & \\ & \mathbf{D}_2^- & -\mathbf{D}_3^+ & & \\ & & \dots & \dots & \\ & & & & (\mathbf{0},\,\mathbf{I})\cdot\mathbf{D}_n^- \end{pmatrix}\cdot\begin{pmatrix} \vec{C}_1 \\ \vec{C}_2 \\ \vec{C}_3 \\ \dots \\ \vec{C}_n \end{pmatrix} \qquad (43)$$

where $\vec{C}_k\in\mathbb{C}^6$, $k=1,\dots,n$ are six-dimensional generally complex vectors containing the unknown coefficients in representation (25); $\mathbf{D}_k^\pm$ are 6x6-matrices defining displacements by Eq. (25) and surface tractions by Eq. (30) on both surfaces of a layer:

$$\mathbf{D}_k^\pm=\begin{pmatrix} \mathbf{m}_{k_1}e^{\pm ir\gamma_{k_1}h_k}\dots \\ \left(\gamma_{k_1}\mathbf{v}\cdot\mathbf{C}_k\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{C}_k\cdot\mathbf{n}\right)\cdot\mathbf{m}_{k_1}e^{\pm ir\gamma_{k_1}h_k}\dots \end{pmatrix} \qquad (44)$$

Despite reported numerical stability [72-74], the comparative study of the original transfer matrix method [58, 59] and the global matrix method [71], revealed that actually both methods exhibit some intrinsic instability resulting in loss of accuracy at high frequencies and large depths of the layers [78]. To overcome the persistent instability, in [78] a more refined rearrangement of the exponential terms appearing in the global matrix method, was suggested.

VI. Cauchy Six-Dimensional Formalism, Homogeneous Plate

That is another variant of a six-dimensional formalism newly developed for analysis of Love and SH waves propagating in monoclinic media [79-81]. Herein, an obvious extension of this formalism to Lamb waves in plates with arbitrary anisotropy is presented. This formalism is mainly based on reduction of the second-order equation (33) to the Cauchy normal form.

For the considered formalism the representation for harmonic Lamb wave is searched in the form (32). By introducing a new vector function

$$\mathbf{v}(x'') = \partial_{x''}\mathbf{f}(x'') \tag{45}$$

equation of motion (26) can be reduced to the Cauchy normal form

$$\partial_{x''}\begin{pmatrix}\mathbf{f}\\ \mathbf{v}\end{pmatrix} = \mathbf{G}\cdot\begin{pmatrix}\mathbf{f}\\ \mathbf{v}\end{pmatrix}, \tag{46}$$

where

$$\mathbf{G} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{A}_1\cdot\mathbf{A}_3 & -\mathbf{A}_1\cdot\left(\mathbf{A}_2 + \mathbf{A}_2^t\right)\end{pmatrix}. \tag{47}$$

and matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are defined by (34). By the analogy with Stroh formalism, 6x6-matrix $\mathbf{G}$ will be called the fundamental matrix.

Similarly to (40), the general solution of Eq. (46) can be represented in Eulerian exponential form

$$\begin{pmatrix}\mathbf{f}(x'')\\ \mathbf{v}(x'')\end{pmatrix} = \exp\left(x''\mathbf{G}\right)\cdot\vec{C}_6, \tag{48}$$

where $\vec{C}_6$ is the six-dimensional generally complex vector of the unknown coefficients defined (up to a multiplier) by the boundary conditions.

The surface traction vector is defined by Eq. (36). Now, combining (48) and (36), the displacements and surface tractions on both sides of the plate take the form

$$\begin{pmatrix}\mathbf{f}(\pm irh)\\ \mathbf{t}_\nu(\pm irh)\end{pmatrix} = \mathbf{Z}\cdot\exp\left(\pm irh\mathbf{G}\right)\cdot\vec{C}_6, \tag{49}$$

where

$$\mathbf{Z} = \begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{A}_2 & \mathbf{A}_1\end{pmatrix}. \tag{50}$$

Note that matrix $\mathbf{Z}$ is non-degenerate due to positive definiteness of the elasticity tensor.

Equation (49) allows us to exclude $\vec{C}_6$ by expressing the displacements and surface tractions on one side of the plate through the corresponding vectors on the other side; that yields

$$\begin{pmatrix}\mathbf{f}(-irh)\\ \mathbf{t}_\nu(-irh)\end{pmatrix} = \mathbf{T}\cdot\begin{pmatrix}\mathbf{f}(+irh)\\ \mathbf{t}_\nu(+irh)\end{pmatrix}, \tag{51}$$

where

$$\mathbf{T} = \mathbf{Z}\cdot\exp\left(-2irh\mathbf{G}\right)\cdot\mathbf{Z}^{-1}. \tag{52}$$

Matrix $\mathbf{T}$ can be considered as the transfer matrix, as it "transfers" displacements and surface tractions from one side of the plate to another. It should be noted that matrix $\mathbf{T}$ is independent of boundary conditions.

Equation (51) is a source for constructing the dispersion relations and obtaining an expression for the limiting velocity $c_{2,\lim}$ at $r\to 0$.

Traction-free plate. If both sides of a plate are traction free, then

$$\mathbf{t}_\nu(\pm irh) = 0. \tag{53}$$

Condition (53) means that a 3×3-mapping

$$R^3 \xrightarrow{(\mathbf{B}_1,\mathbf{B}_2)\cdot\mathbf{T}\cdot\begin{pmatrix}\mathbf{B}_2\\ \mathbf{B}_1\end{pmatrix}} R^3 \tag{54}$$

from three-dimensional space of nontrivial displacements and vanishing surface-tractions on the "upper" surface to the three-dimensional space of the surface-tractions on the "bottom" surface, is degenerate. In (54)

$$\mathbf{B}_1 = \mathbf{0},\quad \mathbf{B}_2 = \mathbf{I}. \tag{55}$$

Matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ are introduced for consistency with other types of boundary conditions that will be considered later in this section.

Condition (54) is equivalent to

$$\det\left(\left(\mathbf{B}_1, \mathbf{B}_2\right)\cdot\mathbf{T}\cdot\begin{pmatrix}\mathbf{B}_2\\ \mathbf{B}_1\end{pmatrix}\right) = 0. \tag{56}$$

Thus, Eq. (56) is the desired dispersion equation for a plate with free boundaries.

Taking in (54) limit at $r\to 0$ does not lead to a meaningful condition, because of vanishing determinant in (56) at $r = 0$ irrespectively of the phase speed. To derive a meaningful equation, consider the first derivative of (54) with respect to the wave number (see [59 - 61]); this yields the following equation for $c_{2,\lim}$

$$\det\left(\left(\mathbf{B}_1, \mathbf{B}_2\right)\cdot\mathbf{Z}\cdot\mathbf{G}\cdot\mathbf{Z}^{-1}\cdot\begin{pmatrix}\mathbf{B}_2\\ \mathbf{B}_1\end{pmatrix}\right) = 0. \tag{57}$$

Remark 4. Substituting matrices (35) into Eq.(57), yields for an isotropic plate Eq.(20).

Clamped plate. If both sides of a plate are clamped, then

$$\mathbf{f}(\pm irh) = 0. \tag{58}$$

Condition (58) means that mapping from the space of vanishing displacements and nontrivial surface tractions on the "upper" surface to the three-dimensional space of the displacements on the "bottom" surface, is degenerate. But, this condition can be expressed by the same equations as Eq. (56) with the only difference in matrices $\mathbf{B}_1$ and $\mathbf{B}_2$:

$$\mathbf{B}_1 = \mathbf{I},\quad \mathbf{B}_2 = \mathbf{0}. \tag{59}$$

Equation for the limiting speed $c_{2,\lim}$ remains the same, as Eq. (57).

Plate with mixed boundary conditions. Herein, two types of mixed boundary conditions at a boundary surface are considered: (i) vanishing normal surface-traction and vanishing tangential displacements; and (ii) vanishing tangential surface-tractions and vanishing normal displacements. Boundary conditions (i) can be written in the form

$$\boldsymbol{\nu}\cdot\mathbf{t}_\nu(\pm irh) = 0,\quad \left(\mathbf{I} - \boldsymbol{\nu}\otimes\boldsymbol{\nu}\right)\cdot\mathbf{f}(\pm irh) = 0. \tag{60}$$

For boundary conditions (60) matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ become

$$\mathbf{B}_1 = \mathbf{I} - \boldsymbol{\nu}\otimes\boldsymbol{\nu},\quad \mathbf{B}_2 = \boldsymbol{\nu}\cdot\mathbf{t}_\nu. \tag{61}$$

Similarly to expressions (60), boundary conditions (ii) can be written in the form

$$\left(\mathbf{I} - \boldsymbol{\nu}\otimes\boldsymbol{\nu}\right)\cdot\mathbf{t}_\nu(\pm irh) = 0,\quad \boldsymbol{\nu}\cdot\mathbf{f}(\pm irh) = 0, \tag{62}$$

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and $\mathbf{B}_1$, $\mathbf{B}_2$ become

$$\mathbf{B}_1 = \boldsymbol{\nu} \cdot \mathbf{t}_{\nu}, \qquad \mathbf{B}_2 = \mathbf{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu}. \tag{63}$$

VII. Cauchy Six-Dimensional Formalism, Multilayered Plate

Consider n-layered plate with the homogeneous anisotropic layers. At the interfaces ideal mechanical contact is assumed.

Transfer matrix method. Assuming that propagation of the Lamb wave within each layer is determined by Eqs. (45)-(50), and writing a sequence of equations (51) that transfers boundary conditions from top to bottom, yield

$$\begin{pmatrix} \mathbf{f}(-irh_n) \\ \mathbf{t}_{\nu}(-irh_n) \end{pmatrix} = \left(\prod_{k=1}^{n} \mathbf{T}_k \right) \cdot \begin{pmatrix} \mathbf{f}(+irh_1) \\ \mathbf{t}_{\nu}(+irh_1) \end{pmatrix}, \tag{64}$$

where the lower index indicates the number of a layer and the transfer matrices $\mathbf{T}_k$ are defined by Eq. (52).

Now, multiplying both sides of Eq. (64) by matrices $\mathbf{B}_1$, $\mathbf{B}_2$, similarly to Eqs. (54) - (56) for the traction-free plate, we arrive at the dispersion equation for the multilayered traction-free plate

$$\det\left((\mathbf{B}_1, \mathbf{B}_2) \cdot \left(\prod_{k=1}^{n} \mathbf{T}_k \right) \cdot \begin{pmatrix} \mathbf{B}_2 \\ \mathbf{B}_1 \end{pmatrix} \right) = 0. \tag{65}$$

Similarly to (54), Eq (65) reflects degeneracy of the mapping

$$R^3 \xrightarrow{(\mathbf{B}_1, \mathbf{B}_2) \cdot \left(\prod_{k=1}^{n} \mathbf{T}_k \right) \cdot \begin{pmatrix} \mathbf{B}_2 \\ \mathbf{B}_1 \end{pmatrix}} R^3 \tag{66}$$

The dispersion equations for other types of boundary conditions have the same form as Eq.(65), but with obvious replacement of matrices $\mathbf{B}_1$ and $\mathbf{B}_2$.

As it was with Eq. (57), taking limit at $r \to 0$ of Eq. (65) does not lead to a meaningful condition, because of vanishing the determinant at $r = 0$ irrespectively of the phase speed. To derive a meaningful equation, consider the first derivative of the mapping (66) with respect to the wave number.

Global matrix method. While the global matrix equation (43) mainly preserves its form, a minor modification is needed to account other types of boundary conditions at the outer boundaries:

$$\vec{\mathbf{0}}_{6n} = \begin{pmatrix} (\mathbf{B}_1, \mathbf{B}_2) \cdot \mathbf{D}_1^+ & & & & \\ & \mathbf{D}_1^- & -\mathbf{D}_2^+ & & \\ & & \mathbf{D}_2^- & -\mathbf{D}_3^+ & \\ & & \dots & \dots & \\ & & & & (\mathbf{B}_1, \mathbf{B}_2) \cdot \mathbf{D}_n^- \end{pmatrix} \cdot \begin{pmatrix} \vec{C}_1 \\ \vec{C}_2 \\ \vec{C}_3 \\ \dots \\ \vec{C}_n \end{pmatrix} \tag{67}$$

Within the considered Cauchy formalism, matrices $\mathbf{D}_k^{\pm}$ change:

$$\mathbf{D}_k^{\pm} = \mathbf{Z}_k \cdot \exp\left(\pm irh_k \mathbf{G}_k \right), \tag{68}$$

where index k indicates the corresponding layer.

VIII. Remarks on Numerical Implementations

Computing matrix exponent. There are two principle methods for computing the matrix exponential [82]. The first one relying on Taylor's series

$$\exp\left(\mathbf{G} x' \right) = \mathbf{I} + \frac{\mathbf{G} x'}{1!} + \frac{\mathbf{G}^2 x'^2}{2!} + \dots \tag{69}$$

is never used in computations due to poor convergence.

Another method relies on reduction of the fundamental matrix to the Jordan normal form. Let the fundamental matrices $\mathbf{G}_{k,\ k=1,\dots,n}$ appearing in (64) and (52) be decomposed into their Jordan normal forms

$$\mathbf{G}_k = \mathbf{W}_k^{-1} \cdot \mathbf{J}_k \cdot \mathbf{W}_k, \tag{70}$$

where $\mathbf{W}_k$ is a matrix composed of eigenvectors and possibly the generalized eigenvectors, while $\mathbf{J}_k$ is a matrix containing eigenvalues and possibly Jordan blocks if $\mathbf{G}_k$ is not a semi simple matrix.

With decomposition (70) the exponential matrices in (52) can be represented in the following form [83, 84]:

$$\exp(-2irh_k \mathbf{G}_k) = \mathbf{W}_k^{-1} \cdot \exp(-2irh_k \mathbf{J}_k) \cdot \mathbf{W}_k. \tag{71}$$

According to [82], representation (71) serves for much faster numerical algorithms than algorithms based on Taylor's expansions.

Improving numerical stability. Performing numerical computations with the exponential matrices leads to numerical instability associated with either loss of precision or overflow due to presence of the exponential terms with large positive powers.

Overflow problem. This problem can be relatively easily solved by a suitable normalization. Let $\left(\lambda_k^{(m)} \right)_{m=1,\dots,6}$ denote (not necessary aliquant) eigenvalues of $\mathbf{G}_k$, and

$$a = \underbrace{\max}_{k,m} \operatorname{Re}\left(-2i\, rh_k\, \lambda_k^{(m)} \right). \tag{72}$$

Multiplying all the exponential matrices in (65) by e^{-a} yields the desired normalization. Indeed, after multiplication, all the exponential terms in (71) will have powers with real parts not exceeding unity.

Loss of precision. As was reported in [78], the extension of the δ-matrix approach [65] to anisotropic media [67-70] and the global matrix methods still manifest numerical instability, which is mainly due to loss of numerical precision. To overcome this problem the following measures are suggested: (i) computing minimal eigenvalue of the resultant matrix (for both transfer and global matrix methods), instead of computing the corresponding determinants; and (ii) use of longer mantissas instead of "double precision" computations.

While being more time-consuming, the first measure allows one to avoid possible overlook of a root. For example, consider a typical case with the following diagonal matrix

$$\mathbf{D} = \operatorname{diag}(\varepsilon, \quad \bar{\varepsilon}, \quad \alpha) \tag{73}$$

where ε is a small (generally complex) eigenvalue associated with the root; and α is a large eigenvalue. In the vicinity of the root the determinant of matrix (73) has no change of sign and thus, can lead to overlook of the root. Moreover, if α increases at approaching the root, the determinant may even increase, at least being not very close to the root.

Necessity to perform multiprecision computations can be demonstrated by considering the following matrix

$$\mathbf{D} = \begin{pmatrix} 1+\varepsilon(t) & -1 \\ -1 & 1-\varepsilon(t) \end{pmatrix}, \tag{74}$$

where $|\varepsilon(t)| < 10^{-15}$ is a small function of the argument t. It is obvious that computing the determinant with double precision accuracy gives zero, nevertheless of the applied numerical algorithm.

In numerical examples considered in [79-81] mantissas from ~25 decimal digits (quadruple precision) to ~1000 digits were used, depending upon the analyzed problem. The multiprecision computations in [79-81] were done by applying algorithms described in [85-88].

Actually, computations with long mantissas revealed that all the numerical problems associated with loss of precision and numerical overflow can be completely eliminated [89].

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