International Journal of Engineering Innovation & Research
Volume 5, Issue 1, ISSN: 2277 – 5668

Time-varying Gain HOSM Control for SISO Uncertain Nonlinear System

Yaozhen Han*[a,b]
[a]State Key Laboratory of Alternate Electrical Power System with Renewable Energy Sources, North China Electric Power University, Beijing, China. email: hyz125@163.com
[b]School of Information Science and Electrical Engineering, Shandong Jiaotong University, Jinan, China

***Abstract* – This paper proposes a time-varying continuous higher-order sliding mode (HOSM) control method for SISO uncertain nonlinear system. HOSM control is converted as finite time stabilization of all-dimension integrator chains with bounded uncertainty. After carried out state feedback control, the auxiliary control law includes two parts. One part achieves finite time convergence of pure integrator chain. The other part, time-varying super-twisting algorithm, is designed to overcome the uncertainties with unknown upper bounds. This control method gets that the control chattering is puny, and over-estimated control gain is nonexistent. Finite time convergence is formally proved based on quadratic form Lyapunov function. Simulation results for an academic example verified the effectiveness of the proposed approach.**

***Keywords* – Continuous sliding mode, Time-varying control gain, Super-twisting algorithm, Robust, Chattering reduction.**

I. Introduction

Sliding mode control (SMC) is a well-known solution for robust control of uncertain nonlinear systems [1]. However, sliding mode chattering, viewed as the most serious problem of conventional SMC [2], can cause high frequency oscillation on actual actuator and restrict its practical application. In view of the chattering magnitude is directly related to control gain, time-varying SMC is also considered as one of the effective way of reducing chattering [3]. However, the fact that the relative degree with respect to sliding variable is one restricts application of conventional sliding mode methods [4].

HOSM control is proposed to remove constraint of relative degree and improve sliding mode accuracy while reserving all the merits of conventional SMC. Among HOSM control, a series of arbitrary-order SMC schemes [5-8] are presented. However, their proofs for finite time stability rely on elusive geometric homogeneity theory. Another class of so-called combinational HOSM control form [9, 10], adopted in this paper, is firstly converted to finite time stabilization problem, then its state feedback control law is divided into nominal control and robust item to establish HOSM. In fact, a more serious drawback for these algorithms [9-10] is that the situation when upper bounds of uncertainties are unknown is not considered. Such upper bounds are often difficult to ascertain in many application systems.

As mentioned earlier, time-varying SMC can weaken sliding mode chattering and reduce the demand on bounds of uncertainties. Adaptive versions of conventional or second-order SMC have achieved good control effect on relevant systems [3, 11]. Very few literatures involve time-varying HOSM control, and they often have such-and-such defects [10, 12]. Actually, another common disadvantage of these results is that the control gain is incapable of decreasing when uncertainties become smaller. It means that the over-estimation of control gain is occurred, and strictly speaking they are not adaptive algorithms.

Adaptive HOSM control schemes are proposed in [[13, 14]. Though finite time stability are formally proved, the control input is not continuous and chattering phenomenon is obvious. In a word, typical problems such as chattering reduction, over-estimation of control gain, finite time convergence and model complexity are not simultaneously solved in existing schemes.

Considering the afore-mentioned, the control objective of this work is to design time-varying continuous HOSM control without over-estimation for a class of nonlinear uncertain SISO system. HOSM control is taken as finite time stabilization of perturbed integrator chains. The usage of nominal continuous control realizes finite time convergence of nominal system, and effectively adjusts transient time. Time-varying super-twisting item eliminates the harmful effect of uncertainties and guarantees system robustness. Numerical example verifies the effectiveness and superiority

The remainder of this work is organized as follows. Section 2 states control problem. In Section 3, the proposed continuous HOSM control is designed and finite time convergence is proved. Section 4 presents the simulation application. Finally some concluding remarks are given in Section 5.

II. Problem Statement

Consider uncertain nonlinear system

$$\begin{cases} \dot{x} = a(x,t) + b(x,t)u \\ y = \sigma \end{cases} \tag{1}$$

where $x \in R^n$, $u \in R$ are state variable and control input respectively. σ is the measurable smooth output vector known as sliding variable. $a(x,t)$ and $b(x,t)$ are uncertain smooth functions whose uncertainties include internal parameter perturbations and external disturbances. It is assumed that the relative degree of system (1) with respect to σ is constant and known, and the associated zero dynamics are stable.

The alleged real higher-order sliding mode are defined as follows.

Definition 1[15]. Consider the nonlinear system (1) and sliding variable $\sigma(x,t)$, provided that $\sigma(x,t)$, $\dot{\sigma}(x,t)$,…, $\sigma^{(r-1)}(x,t)$ are continuous functions. Manifold Ξ_*^r satisfies

$$\Xi_*^r = \{x \,\|\, \sigma(x,t)| < \eta_0(\tau_T), |\dot{\sigma}(x,t)| < \eta_1(\tau_T) \ldots \; |\sigma^{(r-1)}(x,t)| < \eta_{r-1}(\tau_T)\} \tag{3}$$

with $\tau_T > 0$, and $\eta_i \to 0$ when $\tau_T \to 0$, $0 \le i \le r-1$. Assume that the real r^{th}-order sliding set Ξ_*^r is nonempty and is locally an integral set in the Filippov's sense. Then the motion on Ξ^r is called a real r^{th} -order sliding mode with respect to $\sigma(x,t)$.

The control target consists of the finite time convergence of sliding variable $\sigma(x,t)$. When $\sigma(x,t)$ is chosen as tracking error, the goal evolves as finite time tracking of reference signal. The r^{th} time derivative of $\sigma(x,t)$ yields

$$\sigma^{(r)}(x,t) = f(x,t) + g(x,t)u \tag{4}$$

with $f(x,t) = L_f^r\sigma(x) = \sigma^{(r)}(x)|_{u=0}$, $g(x,t) = L_b L_a^{(r-1)}\sigma(x,t)$.

Assumption 1. Smooth functions $f(x,t)$, $g(x,t)$ are uncertain and bounded which can be divided into nominal part and bounded uncertain one, namely

$$\begin{cases} f(x,t) = \bar{f}(x,t) + \Delta f(x,t) \\ g(x,t) = \bar{g}(x,t) + \Delta g(x,t) \end{cases} \tag{5}$$

where $g(x,t)$ >0, $\bar{g}(x,t)$ >0 guaranteeing non singularity of the controllability. There exists unknown upper bounds γ_1 , χ , $\bar{\chi}$ such that formula (6) holds.

$$\begin{cases} \left|\dfrac{\Delta g}{\bar{g}}\right| = \gamma(x,t) \le \gamma_1 < 1 \\ |\bar{f}| \le \bar{\chi} \\ |\Delta f| \le \chi \end{cases} \tag{6}$$

Then r^{th}-order SMC of system (1) with respect to sliding variable σ is equivalently expressed as finite time stabilization control of the following uncertain system

$$\begin{aligned} \dot{z}_i &= z_{i+1} \\ \dot{z}_r &= \bar{f} + \Delta f + (\bar{g} + \Delta g)u \end{aligned} \tag{7}$$

with $z = [z_1, z_2, ..., z_r]^T = [\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}]^T$.

Consider static feedback control for the system (7)

$$u = \frac{1}{\bar{g}}(-\bar{f} + \tau) \tag{8}$$

where τ is the auxiliary control input. This feedback control realizes linearization of system (7) with $\Delta f = 0$ and $\Delta g = 0$. Formulas (7) and (8) will lead to

$$\begin{aligned} \dot{z}_i &= z_{i+1} \\ \dot{z}_r &= \Delta f - \Delta g\bar{g}^{-1}\bar{f} + \left[1 + \Delta g\bar{g}^{-1}\right]\tau \end{aligned} \tag{9}$$

Then the objective of r^{th}-order SMC for system (1) is explicitly converted to finite time stabilization problem of uncertain integrator chains (10).

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= \Delta f - \frac{\Delta g}{\bar{g}}\bar{f} + (1 + \frac{\Delta g}{\bar{g}})\tau \end{aligned} \tag{10}$$

III. Control Design

For stabilizing system (10) in finite time, the virtual control input τ is

$$\tau = \tau_{nom} + \tau_{robt} \tag{11}$$

where nominal control item τ_{nom} impels nominal system converge to origin in finite time, and τ_{robt} conquers uncertainties.

A Nominal Control τ_{nom}

When system uncertainties $\Delta f(x,t) = 0$ and $\Delta g(x,t) = 0$ hold, system (10) is converted to pure chain of integrator

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= \tau_{nom} \end{aligned} \tag{12}$$

This paper will focus on a constructive continuous finite time control law presented in [16] for all-dimension integrator chain.

Theorem 1 [16]. Choose positive constants $c_1,...,c_r$, $c_1^*,...,c_r^*$ to make $p^r + c_r p^{r-1} + \cdots + c_2 p + c_1$ and $p^r + c_r^* p^{r-1} + \cdots + c_2^* p + c_1^*$ be *Hurwitz* polynomial respectively. Then there exists $\hat{\lambda} \in (0,1)$, for any $\ell \in (1-\hat{\lambda}, 1)$, system states in (12) converge to origin in finite time under control law

$$\begin{aligned} \tau_{nom} = &-c_1 sign(z_1)|z_1|^{\ell_1} - \cdots - c_r sign(z_r)|z_r|^{\ell_r} \\ &- \varepsilon_1 z_1 - \cdots - \varepsilon_r z_r \end{aligned} \tag{13}$$

with $\ell_{i-1} = \dfrac{\ell_i \ell_{i+1}}{2\ell_{i+1} - \ell_i}$, $i \in \{2,...,r\}$, $\ell_{r+1} = 1$, $\ell_r = \ell$.

B Time-varying Continuous Robust Control

As mentioned above, Finite time stability cannot be guaranteed when existing uncertain items in system (9). This section proposes a time-varying continuous robust control to counteract uncertainties with unknown upper bounds based on adaptive supper-twisting algorithm. Then system states can converge to arbitrarily small neighborhood of origin.

International Journal of Engineering Innovation & Research
Volume 5, Issue 1, ISSN: 2277 – 5668

Define a new sliding surface s

$$\begin{cases} s = z_r + \tau_{aux} \\ \dot{\tau}_{aux} = -\tau_{nom} \end{cases} \quad (14)$$

Calculating the time derivative of s and in view of formula (10)

$$\begin{aligned} \dot{s} = \dot{z}_r + \dot{\tau}_{aux} &= \Delta f - \frac{\Delta g}{g}\bar{f} + (1+\frac{\Delta g}{g})\tau - \tau_{nom} \\ &= \Delta f + \frac{\Delta g}{g}(\tau_{nom} + \tau_{robt} - \bar{f}) + \tau_{robt} \\ &= \tilde{\psi}_1 + (1+\frac{\Delta g}{g})\tau_{robt} \\ &= \tilde{\psi}_1 + \beta_1\tau_{robt} \end{aligned} \quad (15)$$

where $\tilde{\psi}_1 = \Delta f + \frac{\Delta g}{g}(\tau_{nom} - \bar{f})$, $\beta_1 = 1 + \frac{\Delta g}{g}$.

Assumption 3. $|\dot{\tilde{\psi}}_1| \le \tilde{\psi}_{1d}$. $\tilde{\psi}_{1d}$ is the unknown upper bound.

The objective is to design continuous control law without over-estimation of control gain to establish real second-order sliding mode with respect to s, and then real r^{th}-order sliding mode with respect to σ . Consider adaptive super-twisting control law

$$\begin{cases} \tau_{robt} = -k_1 |s|^{\frac{1}{2}} sign(s) + \tau_{robt1} \\ \dot{\tau}_{robt1} = -\frac{k_2}{2} sign(s) \end{cases} \quad (16)$$

where k_1 and k_2 are adaptive control gains varying with γ_1 and $\tilde{\psi}_{1d}$.

It is assumed that β_1 is an uncertain piece-wise constant. The system (15) and (16) can be rephrased as

$$\begin{cases} \dot{s} = -k_1\beta_1 |s|^{\frac{1}{2}} sign(s) + \alpha_* \\ \dot{\alpha}_* = -\frac{k_2\beta_1}{2} sign(s) + \dot{\tilde{\psi}}_1 + \dot{\beta}_1\tau_{robt1} \\ \alpha_*(0) = 0 \\ \alpha_* = \tilde{\psi}_1 + \beta_1\tau_{robt1} \end{cases} \quad (17)$$

Suppose that $\dot{\beta}_1\tau_{rob1}$ has unknown bound $\gamma_3 > 0$, that is

$$|\dot{\beta}_1\tau_{robt1}| \le \tfrac{1}{2}|\dot{\beta}_1| \int_0^t k_2 ds \le \gamma_3 \quad (18)$$

Considering the time varying gain k_2 is bounded, i.e. $|k_2| \le k^*$, $k^* > 0$ (this will be proved later). Then

$$|\dot{\beta}_1\tau_{robt1}| \le \tfrac{1}{2}|\dot{\beta}_1| k^* t \le \gamma_3 \quad (19)$$

The bound of the uncertain function $\xi = \dot{\psi}_1 + \dot{\beta}_1\tau_{robt1}$ is existing and unknown, i.e.

$$\dot{\xi} \le \tilde{\psi}_{1d} + \gamma_3 = \gamma_4 \quad (20)$$

The design of τ_{robt} is evolved into gain-scheduled super-twisting SMC for system (17) under unknown parameters γ_1 , γ_4 , $\tilde{\psi}_{1d}$. To dynamically increase k_1 and k_2 until the establishment of second-order sliding mode, then the gains begin to decrease. Once s or $\dot{s}$ deviates from $s = \dot{s} = 0$, the gain increasement will impel s , $\dot{s} \to 0$ in finite time.

Theorem 2. Consider nonlinear system (1), suppose that Assumption 1-Assumption 3 are all fulfilled. γ_1 , γ_4 and $\tilde{\psi}_{1d}$ are existing and unknown. Then control law

$$u = \frac{1}{g}(-\bar{f} + \tau_{nom} + \tau_{robt}) \quad (21)$$

ensure establishment of real r^{th}-order sliding mode with respect to σ in finite time. τ_{nom} and τ_{robt} are determined by (13) and (16), and the adaptive law of (16) is

$$\begin{cases} \dot{k}_1 = \begin{cases} \delta_1\sqrt{\frac{\upsilon_1}{2}} sign(|s| - \eta_0) & \text{if } k_1 > k_m \\ \varsigma & \text{if } k_1 \le k_m \end{cases} \\ k_2 = 2\omega k_1 \end{cases} \quad (22)$$

where δ_1 , υ_1 , ς are arbitrary positive constants, k_m is arbitrarily small positive constant, and η_0 is the critical variable of establishment for real sliding mode.

Proof

Firstly, system (17) is formulated as the form which is convenient to analyze under Lyapunov approach. To introduce a new state variable q

$$q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T = \begin{bmatrix} |s|^{\frac{1}{2}} sign(s) & \alpha_* \end{bmatrix}^T \quad (23)$$

Then system (17) can be represented as

$$\begin{cases} \dot{q}_1 = \frac{1}{2|q_1|}(-k_1\beta_1 q_1 + q_2) \\ \dot{q}_2 = -\frac{k_2\beta_1}{2|q_1|} q_1 + \dot{\xi} \end{cases} \quad (24)$$

and the matrix form of (24) is

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \frac{1}{2|q_1|}\begin{bmatrix} -k_1\beta_1 & 1 \\ -k_2\beta_1 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \frac{1}{2|q_1|}\begin{bmatrix} 1 & 0 \\ 0 & 2|q_1| \end{bmatrix}\begin{bmatrix} 0 \\ \dot{\xi} \end{bmatrix} \quad (25)$$

Consider Assumption 3 and formula (20)

$$\dot{\xi} = \frac{\rho_1}{2} sign(s) = \frac{\rho_1}{2}\frac{q_1}{|q_1|} \quad (26)$$

with $0 < \rho_1 < 2\gamma_4$.

Take note of (26), then (25) can be rewritten as

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \bar{A}(q_1)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad (27)$$

where $\bar{A}(q_1) = \frac{1}{2|q_1|}\begin{bmatrix} -k_1\beta_1 & 1 \\ -k_2\beta_1 + \rho_1 & 0 \end{bmatrix}$.

Actually, it is observed that $|q_1| = |s|^{1/2}$, $sign(q_1) = sign(s)$, s , $\dot{s} \to 0$ is set up in finite time. Then let's analyze stability for system (25). Choosing Lyapunov function

International Journal of Engineering Innovation & Research
Volume 5, Issue 1, ISSN: 2277 – 5668

$$V(q_1,q_2,k_1,k_2)=V_0+\frac{1}{2\upsilon_1}(k_1-k_1^*)^2+\frac{1}{2\upsilon_2}(k_2-k_2^*)^2 \quad (28)$$

where $V_0=(\chi+4\omega^2)q_1^2+q_2^2-4\omega q_1q_2=q^T\Gamma q$, $\Gamma=\begin{bmatrix}\chi+4\omega^2 & -2\omega\\ -2\omega & 1\end{bmatrix}$, $\chi>0$, k_1^*, k_2^* are some constants.

It is easy to verify that Γ is positive definite if $\chi>0$ and ω are any real number. Derivative of (28) is deduced as

$$\dot V(q_1,q_2,k_1,k_2)=q^T\left[\bar A^T(q_1)\Gamma+\Gamma\bar A(q_1)\right]q+\frac{1}{\upsilon_1}(k_1-k_1^*)\dot k_1+\frac{1}{\upsilon_2}(k_2-k_2^*)\dot k_2 \quad (29)$$

Take note of (25) and (27), then the first item of (29) is

$$\dot V_0=q^T\left[\bar A^T(q_1)\Gamma+\Gamma\bar A(q_1)\right]q\le-\frac{1}{2|q_1|}q^T\Theta q \quad (30)$$

In view of Assumption 3 and formula (26), the symmetric matrix Θ is

$$\Theta=\begin{bmatrix}\Theta_{11} & \Theta_{12}\\ \Theta_{21} & 4\omega\end{bmatrix} \quad (31)$$

where $\Theta_{11}=2\chi k_1\beta_1+4\omega\beta_1(2\omega k_1-k_2)+4\omega\rho_1$, $\Theta_{12}=\Theta_{21}=(k_2\beta_1-2\omega k_1\beta_1-\chi-4\omega^2)-\rho_1$.

For the positive definiteness of Θ, to define

$$k_2=2\omega k_1 \quad (32)$$

when k_1 satisfies

$$k_1>\frac{(2\gamma_4+\chi+4\omega^2)^2}{12\omega\chi(1-\upsilon_1)}-\frac{\omega(4\gamma_4+1)}{\chi(1-\upsilon_1)} \quad (33)$$

Θ is positive definite and minimum eigen value $\lambda_{\min}(\Theta)\ge 2\omega$.

Take into account (30) and suppose (32), (33) are fulfilled, because

$$\dot V_0\le-\frac{1}{2|q_1|}q^T\Theta q\le-\frac{2\omega}{2|q_1|}q^Tq=-\frac{\omega}{|q_1|}\|q\|^2 \quad (34)$$

$$\lambda_{\min}(\Gamma)\|q\|^2\le q^T\Gamma q\le\lambda_{\max}(\Gamma)\|q\|^2 \quad (35)$$

where $\|q\|^2=q_1^2+q_2^2=|s|+q_2^2$ and

$$|q_1|=|s|^{\frac12}\le\|q\|\le\frac{V_0^{\frac12}}{\lambda_{\min}^{\frac12}(\Gamma)} \quad (36)$$

Then

$$\dot V_0\le-\rho V_0^{\frac12} \quad (37)$$

with $\rho=\frac{\omega\lambda_{\min}^{1/2}(\Gamma)}{\lambda_{\max}(\Gamma)}$.

Now, take note of (29) , (34), and $\lambda_1=k_1-k_1^*$ $\lambda_2=k_2-k_2^*$, then

$$\begin{aligned}\dot V(q,k_1,k_2)&=\dot q^T\Gamma q+q^T\Gamma\dot q+\frac{1}{\upsilon_1}\lambda_1\dot k_1+\frac{1}{\upsilon_2}\lambda_2\dot k_2\\ &\le-\frac{1}{|q_1|}q^T\Theta q+\frac{1}{\upsilon_1}\lambda_1\dot k_1+\frac{1}{\upsilon_2}\lambda_2\dot k_2\\ &\le-\rho V_0^{\frac12}+\frac{1}{\upsilon_1}\lambda_1\dot k_1+\frac{1}{\upsilon_2}\lambda_2\dot k_2\\ &=-\rho V_0^{\frac12}-\frac{\delta_1}{\sqrt{2\upsilon_1}}|\lambda_1|-\frac{\delta_2}{\sqrt{2\upsilon_2}}|\lambda_2|+\frac{1}{\upsilon_1}\lambda_1\dot k_1+\frac{1}{\upsilon_2}\lambda_2\dot k_2\\ &\quad+\frac{\delta_1}{\sqrt{2\upsilon_1}}|\lambda_1|+\frac{\delta_2}{\sqrt{2\upsilon_2}}|\lambda_2|\end{aligned} \quad (38)$$

Consider the well-known in equation $(a^2+b^2+c^2)^{\frac12}\le|a|+|b|+|c|$ and formula (28), it is deduced as

$$-\rho V_0^{\frac12}-\frac{\delta_1}{\sqrt{2\upsilon_1}}|\lambda_1|-\frac{\delta_2}{\sqrt{2\upsilon_2}}|\lambda_2|\le-\upsilon_0\sqrt{V(q,k_1,k_2)} \quad (39)$$

where $\upsilon_0=\min(\rho,\delta_1,\delta_2)$. Take note of (39), (38) can be rewritten as

$$\dot V(q_1,q_2,k_1,k_2)\le-\upsilon_0\sqrt{V(q_1,q_2,k_1,k_2)}+\frac{1}{\upsilon_1}\lambda_1\dot k_1+\frac{1}{\upsilon_2}\lambda_2\dot k_2+\frac{\delta_1}{\sqrt{2\upsilon_1}}|\lambda_1|+\frac{\delta_2}{\sqrt{2\upsilon_2}}|\lambda_2| \quad (40)$$

Let's suppose adaptive gains k_1, k_2 be bounded which will be proved later. Then positive constants k_1^* and k_2^* are existing such that $k_1-k_1^*<0$, $k_2-k_2^*<0$. Consider the above assumption. (40) can be deduced as

$$\dot V(q_1,q_2,k_1,k_2)\le-\upsilon_0\left[V(q_1,q_2,k_1,k_2)\right]^{\frac12}-|\lambda_1|\left(\frac{1}{\upsilon_1}\dot k_1-\frac{\delta_1}{\sqrt{2\upsilon_1}}\right)-|\lambda_2|\left(\frac{1}{\upsilon_2}\dot k_2-\frac{\delta_2}{\sqrt{2\upsilon_2}}\right) \quad (41)$$

and then

$$\dot V(q_1,q_2,k_1,k_2)\le-\upsilon_0\left[V(q_1,q_2,k_1,k_2)\right]^{\frac12}+\zeta \quad (42)$$

where

$$\zeta=-|\lambda_1|\left(\frac{1}{\upsilon_1}\dot k_1-\frac{\delta_1}{\sqrt{2\upsilon_1}}\right)-|\lambda_2|\left(\frac{1}{\upsilon_2}\dot k_2-\frac{\delta_2}{\sqrt{2\upsilon_2}}\right) \quad (43)$$

S_1. It is assumed that $|s|>\eta_0$ and $k_1>k_m$, then, according to (22)

$$\dot k_1=\delta_1\sqrt{\frac{\upsilon_1}{2}} \quad (44)$$

$$\zeta=-|\lambda_2|\left(\frac{1}{\upsilon_2}\dot k_2-\frac{\delta_2}{\sqrt{2\upsilon_2}}\right) \quad (45)$$

To choose $\omega=\frac{\delta_2}{2\delta_1}\sqrt{\frac{\upsilon_2}{\upsilon_1}}$, then according to (32)

$$\dot k_2=2\omega\dot k_1=\omega\delta_1\sqrt{2\upsilon_1}=\delta_2\sqrt{\frac{\upsilon_2}{2}} \quad (46)$$

Take into account (46), $\zeta = 0$ in (45) holds, and

$$\dot{V}(q_1, q_2, k_1, k_2) \le -\upsilon_0 [V(q_1, q_2, k_1, k_2)]^{\frac{1}{2}} \quad (47)$$

It is notable that k_1 must satisfy with (33) for finite time convergence. This implies k_1 will increase according to (44) until (33) satisfies that Θ is positive definite and (47) is effective. Then, according to (47) and take note of definition of s, it is obvious $|s| \le \eta_0$ holds in finite time.

S_2. To suppose $|s| < \eta_0$, then k_1 satisfies

$$\dot{k}_1 = \begin{cases} -\delta_1 \sqrt{\frac{\upsilon_1}{2}} & \text{if } k_1 > k_m \\ \varsigma & \text{if } k_1 \le k_m \end{cases} \quad (48)$$

and the term

$$\zeta = \begin{cases} 2|k_1 - k_1^*| \frac{\delta_1}{\sqrt{2\upsilon_1}} & \text{if } k_1 > k_m \\ -|k_m - k_1^* + \varsigma.t| (\frac{\varsigma}{\upsilon_1} - \frac{\delta_1}{\sqrt{2\upsilon_1}}) & \text{if } k_1 \le k_m \end{cases} \quad (49)$$

become positive. It is notable that the second item of (49) is effective just in finite time because the value k_1 will instantly increase as long as $k_1 \le k_m$. Hence $k_1 = k_m + \varsigma.t$, then the first item of (49) become effective.

According to (49), the symbol of differential of V become uncertain. $|s|$ may become greater than η_0 due to the decrease of control gain. The condition defined in S_1 is set up provided $|s|$ become greater than η_0, and then s will enter into $|s| \le \eta_0$ in finite time. Accordingly, s can stay in a bigger region, that is a real sliding mode $|s| \le \eta_1$, $\eta_1 > \eta_0$. Within $|s| \le \eta_0$, the value of $|\dot{s}|$ can be estimated by (17) and (22)

$$|\dot{s}| \le (1-\upsilon_1) k_1 \eta_0^{\frac{1}{2}} + [\omega(1-\upsilon_1)k_1 + \gamma_4](t_2 - t_1) = \overline{\eta}_2 \quad (50)$$

where t_1 and t_2 are the time that s enter into and leave $|s| \le \eta_0$ respectively. After $\eta_0 \le |s| \le \eta_1$ holds, then

$$|\dot{s}| \le (1+\upsilon_1)(\mu_1^{1/2} + \varepsilon)(k_1 + \omega_1 \sqrt{\frac{\mu_1 \upsilon_1}{2}})(t_3 - t_2) + \delta_4 (t_3 - t_2) = \tilde{\eta}_2 \quad (51)$$

where t_2, t_3 are the time that s leave and enter into $|s| \le \mu_1$ respectively. Combining (50) with (51), then

$$|\dot{s}| \le \max(\overline{\eta}_2, \tilde{\eta}_2) = \eta_2 \quad (52)$$

Now, it is proved the establishment of real second sliding mode in finite time with respect to s. In actual sliding mode, the equivalent control of τ_{robt} is defined as τ_{robt}^{eq} when $\dot{s} = \eta_2$, it is deduced from (16)

$$\tau_{rob}^{eq} = \beta_1^{-1}(\eta_2 - \tilde{\psi}_2) \quad (53)$$

Take $\tau = \tau_{nom} + \tau_{robt}^{eq}$ into formula (10), the equivalent closed-loop dynamics similar to nominal system (12) is acquired. Because τ_{nom} is designed according to Theorem 1, system states can converge to the neighborhood of origin. Take note of definition of s, the real r^{th}-order sliding mode with respect to σ is established in finite time. Theorem 2 is proved.

IV. Case Study

To expediently illustrate the effectiveness of the controller in Theorem 2, an academic dynamic is taken as the controlled system

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = 10 + \Delta_1 + (2 + \Delta_2) u \end{cases} \quad (54)$$

where Δ_1 and Δ_2 are system uncertainties, and the initial states are chosen as $z_1(0) = 3$, $z_2(0) = 0$, $z_3(0) = -2$.

The feedback control (8) is chosen as $u = \frac{1}{2}(-10 + \tau)$, which leads to

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = \Delta_1 - 5\Delta_2 + \frac{1}{2}\Delta_2 \tau + \tau \end{cases} \quad (55)$$

System (15) becomes pure triple integrator chain when $\Delta_1 = 0$ and $\Delta_2 = 0$. According to Theorem 1, parameters are chosen as $\ell = \frac{3}{4}$, $c_1 = 1$, $c_2 = \frac{3}{2}$, $c_3 = \frac{3}{2}$, $c_1^* = \frac{1}{3}$, $c_2^* = 1$, $c_3^* = \frac{1}{2}$, then the nominal control law is

$$\tau_{nom} = -sign(z_1)|z_1|^{1/2} - \frac{3}{2} sign(z_2)|z_2|^{3/5} - \frac{3}{2} sign(z_3)|z_3|^{3/4} - \frac{1}{3} z_1 - z_2 - \frac{1}{2} z_3 \quad (56)$$

For comparison, the homogeneous control law in [17] is designed as

$$\tau_{nom} = -sign(z_1)|z_1|^{1/2} - \frac{3}{2} sign(z_2)|z_2|^{3/5} - \frac{3}{2} sign(z_3)|z_3|^{3/4} \quad (57)$$

The simulation solver is based on Euler's integral method and the simulation step size is 1ms.

The uncertainties Δ_1 and are chosen as

$$\Delta_1 = 2\sin(0.1t), \ \Delta_2 = \cos(t) \quad (58)$$

and their differential are bounded and unknown, here they are chosen as (58): $\Delta_1 = 2\sin(0.1t)$, $\Delta_2 = \cos(t)$. This satisfies the three assumptions. According to Theorem 2, the nominal control τ_{nom} is still chosen as formula (56), and the parameters are $k_m = 3$, $\delta_1 = 1$, $\upsilon_1 = 2$, $\varsigma = 5$, $\omega = 0.55$, $\eta_0 = 0.001$.

$$\begin{cases} \dot{k}_1 = \begin{cases} 1\sqrt{\dfrac{2}{2}}sign(s-0.001) & if\ \ k_1 > 10 \\ 5 & if\ \ k_1 \le 10 \end{cases} \\ k_2 = 2*0.55*k_1 \end{cases} \quad (59)$$

For highlighting advantages of the proposed controller, the controller in [14] is also designed and executed. The controller is

$$\begin{cases} \tau = \tau_1 + \tau_2 \\ \tau_1 = -10\dfrac{z_3 + 2(|z_2| + |z_1|^{2/3})^{-1/2}(z_2 + |z_1|^{2/3} sign(z_1))}{|z_3| + 2(|z_2| + |z_1|^{2/3})^{1/2}} \\ \tau_2 = -K(t) sign(z_3 + \tau_{aux}) \\ \dot{\tau}_{aux} = -\tau_1 \\ K(t) = \begin{cases} 10^{-1}|\delta_3|^{\frac{1}{4}} sign(\delta_3) & \text{if} \quad K(t) > 2 \\ 5 & \text{if} \quad K(t) \le 2 \end{cases} \\ \delta_3 = |z_1| + 10^{-3}|z_2| + 10^{-6}|z_3| - 10^{-8} \end{cases} \quad (60)$$

Fig.1 and Fig.2 show the states and control input under the proposed algorithm. Fig.1 also demonstrates that real third-order sliding mode is established in finite time. Partial enlarged detail displays that there exists periodically loss of third-order sliding mode, due to the periodic uncertainties Δ_1 and Δ_2. Continuity of control input is shown in Fig.2 and the control chattering is greatly diminished. Fig. 3 is the time-varying control gain k_1 according to the uncertainties, and it has the same trend for k_2.

Simulation results under the controller in [14] are shown in Fig.4, Fig.5 and Fig.6. The control gain is adaptive as shown in Fig.6. Fig.4 shows the states can track the initial values at the beginning and can converge to neighborhood of origin in finite time. However, the transient process is long and overshoot is serious. What is more serious, the chattering is critical and could not be applied for real system actually.

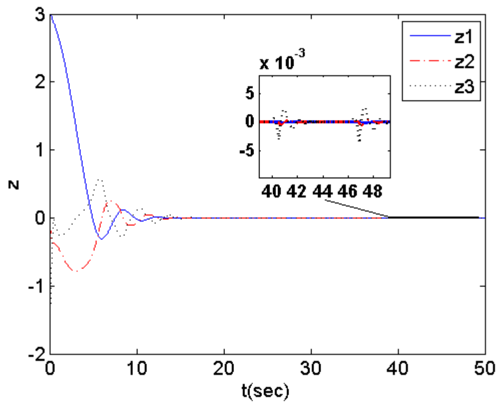

Fig.1. States

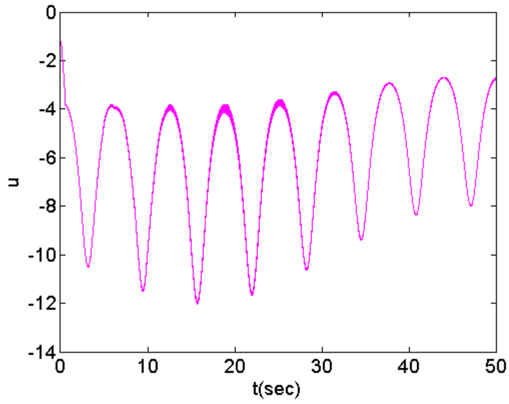

Fig. 2. Control input

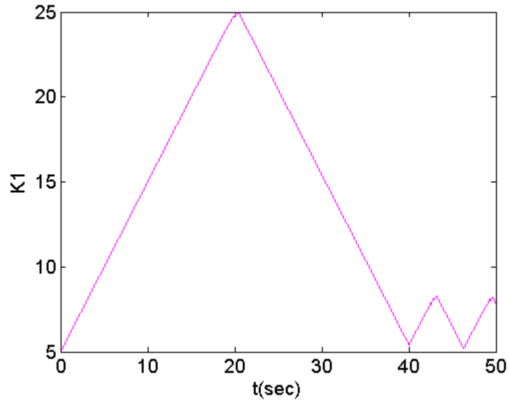

Fig. 3. Adaptive gain k_1

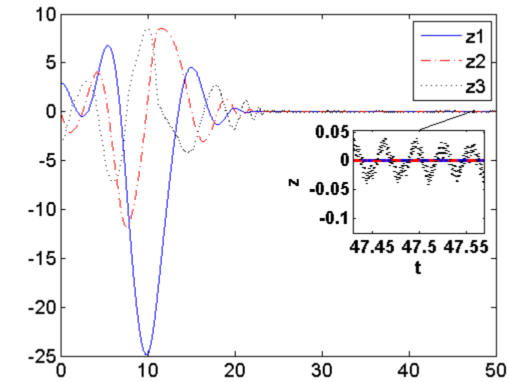

Fig. 4. States

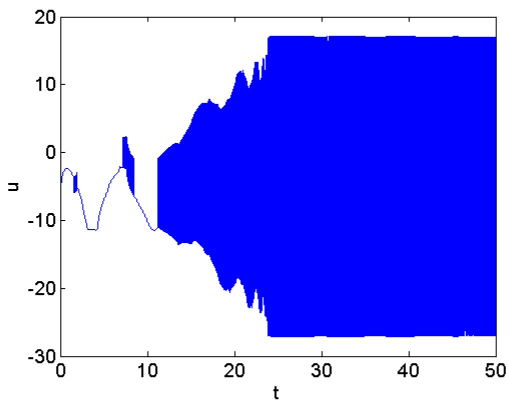

Fig. 5. Control input

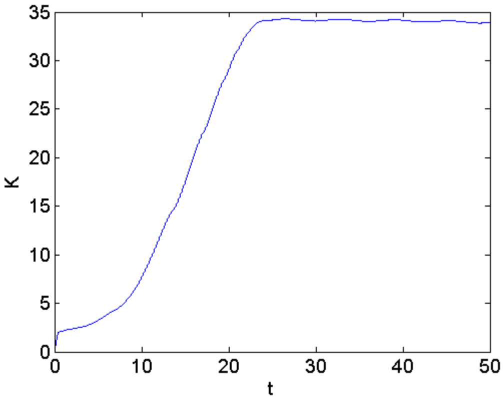

Fig. 6. Adaptive gain

V. Conclusion

A gain-scheduled continuous robust control scheme based on higher-order sliding mode is proposed in this paper. The controller comprises a nominal part that stabilizes the system without uncertainties, together with a gain-scheduled super-twisting part to guarantee robustness for a class of uncertainties with unknown upper bound. The real higher-order sliding mode without over-estimated control gain is established, and finite time stability of closed-loop system is formally proved. Simulation results illustrate effectiveness of the proposed strategy.

VI. Acknowledgments

This work was supported by National Natural Science Foundation of China under Grant 61273144; Shandong Provincial Natural Science Foundation of China under Grant No. ZR2013EEL014, No. ZR2013ZEM006, ZR2014FM033; Application Fundamental Research Project of Ministry of Transport of China (No. 2014329817130); and Project of the Communications Department of Shandong province (No. 2013A16-04).

References

[1] Khanesar, Mojtaba Ahmadieh, et al. "Adaptive Indirect Fuzzy Sliding Mode Controller for Networked Control Systems Subject to Time-Varying Network-Induced Time Delay." *Fuzzy Systems, IEEE Transactions on* 23.1 (2015): 205-214.

[2] Li, Hongyi, et al. "Adaptive sliding-mode control for nonlinear active suspension vehicle systems using T–S fuzzy approach." *Industrial Electronics, IEEE Transactions on* 60.8 (2013): 3328-3338.

[3] Plestan, Franck, et al. "New methodologies for adaptive sliding mode control." *International journal of control* 83.9 (2010): 1907-1919.

[4] Tang, W. Q., and Y. L. Cai. "High‐order sliding mode control design based on adaptive terminal sliding mode." *International Journal of Robust and Nonlinear Control* 23.2 (2013): 149-166.

[5] Levant, Arie. "Homogeneity approach to high-order sliding mode design."*Automatica* 41.5 (2005): 823-830.

[6] Levant, Arie. "Quasi-continuous high-order sliding-mode controllers."*Automatic Control, IEEE Transactions on* 50.11 (2005): 1812-1816.

[7] Levant, Arie, and Lela Alelishvili. "Integral high-order sliding modes." *IEEE Transactions on Automatic Control* 52.7 (2007): 1278.

[8] Levant, Arie, and Alon Michael. "Adjustment of high‐order sliding‐mode controllers." *International Journal of Robust and Nonlinear Control* 19.15 (2009): 1657-1672.

[9] Laghrouche, Salah, Franck Plestan, and Alain Glumineau. "Higher order sliding mode control based on integral sliding mode." *Automatica* 43.3 (2007): 531-537.

[10] Defoort, Michael, et al. "A novel higher order sliding mode control scheme."*Systems & Control Letters* 58.2 (2009): 102-108.

[11] Shtessel, Yuri, Mohammed Taleb, and Franck Plestan. "A novel adaptive-gain supertwisting sliding mode controller: methodology and application."*Automatica* 48.5 (2012): 759-769.

[12] Zong, Qun, Z-S. Zhao, and Juyong Zhang. "Higher order sliding mode control with self-tuning law based on integral sliding mode." *IET control theory & applications* 4.7 (2010): 1282-1289.

[13] Wang, Liang, Yongzhi Sheng, and Xiangdong Liu. "A novel adaptive high-order sliding mode control based on integral sliding mode." *International Journal of Control, Automation and Systems* 12.3 (2014): 459-472.

[14] Taleb, Mohammed, Franck Plestan, and B. Bououlid. "An adaptive solution for robust control based on integral high‐order sliding mode concept."*International Journal of Robust and Nonlinear Control* 25.8 (2015): 1201-1213.

[15] Levant, Arie. "Higher-order sliding modes, differentiation and output-feedback control." *International journal of Control* 76.9-10 (2003): 924-941.

[16] Shtessel Y, Kochalummoottil J, Edwards C, et al, "Continuous adaptive finite reaching time control and second-order sliding modes," *IMA Journal of Mathematical Control and Information* 30(1) (2013) 97-113.

[17] Bhat, Sanjay P., and Dennis S. Bernstein. "Geometric homogeneity with applications to finite-time stability." *Mathematics of Control, Signals and Systems* 17.2 (2005): 101-127.

Author's Profile

Yaozhen Han
He received the B.S. degree in automation in 2005 from Qingdao University of Science and Technology, Qingdao, China, the M.S. degree in automatic control in 2008, from Shandong University, Jinan, China. He is currently working toward the Ph.D. degree in automatic control at North China Electric Power University, Beijing, China. Now he also works as an associate professor in Shandong Jiaotong University, Jinan, China. His research interests include modeling, optimization and sliding mode control of power plants.