Root System and Dynkin Diagrams for the General Class of Indefinite Quasi Affine Kac Moody Algebras $QAG_2^{(1)}$

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***Abstract* – In this paper the general class of indefinite quasi affine Kac Moody algebras $QAG_2^{(1)}$ are considered. The root system of this family is studied; in particular, the short and long real roots are obtained and some of the basic properties of imaginary roots are studied for a particular class. This class is found to satisfy the purely imaginary property, i.e. all the imaginary roots are purely imaginary; isotropic and minimal imaginary roots are computed for a particular class of quasi affine Kac Moody algebras belonging to the family $QAG_2^{(1)}$. The complete set of connected, non isomorphic Dynkin diagrams of $QAG_2^{(1)}$, where an extended vertex is connected to either one, two or all three vertices of the base affine family $G_2^{(1)}$ is given. This classification is the generalization of the one obtained earlier, wherein, only the case where the extended vertex of $QAG_2^{(1)}$ is connected to all the three vertices of the affine family $G_2^{(1)}$. Among this set of diagrams, the extended hyperbolic type and diagrams of indefinite, non extended hyperbolic type are identified.**

***Keywords* – Dynkin Diagrams, Imaginary Roots, Kac Moody Algebras, Quasi Affine.**

I. Introduction

The theory of Kac Moody algebras was developed simultaneously and independently by Kac and Moody in 1968 ([5],[9]). Kac Moody algebras are broadly classified into finite, affine and indefinite types. A lot of research has already been carried out in the finite and affine cases, with interesting applications to various branches of Mathematics and Physics; the indefinite type of Kac Moody algebras are yet to be understood completely. Complete classification of Dynkin diagrams for the finite, affine and, hyperbolic types are already given ([5], [25]). Also the characterization of the root systems of the finite and affine types have been obtained ([5],[9],[25]). Strictly imaginary roots and special imaginary roots were studied by Casperson [3] and Bennett [2]. Sthanumoorthy and Uma Maheswari introduced Purely imaginary roots ([10],[11]). While classifying the Kac Moody algebras with the purely imaginary property, Sthanumoorthy and Uma Maheswari introduced a class of indefinite type namely, the extended hyperbolic, which are natural extensions of the hyperbolic type and determined the structure and computed root multiplicities up to level 5 for $EHA_1^{(1)}$, $EHA_2^{(2)}$([12]-[14]).

In the indefinite Kac Moody algebras, hyperbolic Kac Moody algebras were studied by Feingold & Frenkel [4], Benkart, Kang & Misra ([1]), Kang ([6]-[8]); Uma Maheswari introduced indefinite Quasi hyperbolic and indefinite Quasi affine Kac Moody algebras ([15],[18]). Uma Maheswari and et.al studied Quasi hyperbolic Kac Moody algebras, namely QHG2, $QHA_2^{(1)}$, $QHA_4^{(2)}$, $QHA_7^{(2)}$, $QHA_5^{(2)}$ in ([20]-[24]), about the quasi hyperbolic Dynkin diagrams of rank 3 in [16] and the indefinite quasi affine Kac Moody families $QAD_3^{(2)}$, $QAG_2^{(1)}$ in ([17]-[19]). The family $QAG_2^{(1)}$ was already defined in [18], where the structure of a specific class of $QAG_2^{(1)}$ was determined using homological techniques and spectral sequences theory. In [18], the classification of Dynkin diagrams was also given for a particular class, where the extended fourth vertex of $QAG_2^{(1)}$ was connected to all the three vertices of the affine family $G_2^{(1)}$.

In this paper, we consider the more general class of $QAG_2^{(1)}$, obtained from the affine family $G_2^{(1)}$. We give the complete classification of non isomorphic connected Dynkin diagrams (999 in number) associated with $QAG_2^{(1)}$; while 729 such Dynkin diagrams were already given in [18], now we explicitly give the remaining 270 non isomorphic connected Dynkin diagrams, where the extended vertex is connected to either one or any two vertices of the base affine family $G_2^{(1)}$. A particular class of Kac Moody algebras belonging to $QAG_2^{(1)}$ is considered; the basic properties of real and imaginary roots for $QAG_2^{(1)}$ are also studied.

II. Preliminaries

The detailed study on Kac Moody algebras can be referred from Kac[5] and Wan[25].

Definition 1: [5], [25] A realization of a matrix $A=(a_{ij})_{i,j=1}^{n}$ is a triple $(H,\Pi,\Pi^{\vee})$ where l is the rank of A, H is a 2n - l dimensional complex vector space, $\Pi=\{\alpha_1,\alpha_2,...,\alpha_n\}$ and $\Pi^{\vee}=\{\alpha_1^{\vee},\alpha_2^{\vee},...,\alpha_n^{\vee}\}$ are linearly independent subsets of H* and H respectively, satisfying $\alpha_j(\alpha_i^{\vee})=a_{ij}$ for i, j = 1,….,n. Π is called the root basis. Elements of Π are called simple roots. The root lattice generated by Π is $Q=\sum_{i=1}^{n} z\alpha_i$.

The Kac-Moody algebra g(A) associated with a GCM $A=(a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements ei, fi, i=1,2,…n and H with the following defining relations :

$[h,h']=0, h,h'\in H, [e_i, f_j]=\delta_{ij}\alpha_i^v$
$[h,e_j]=\alpha_j(h)e_j, [h,f_j]=-\alpha_j(h)f_j,$
$(ade_i)^{1-a_{ij}}e_j=0, (adf_i)^{1-a_{ij}}f_j=0,$
$\forall\ i\neq j,\ i,j\in N$

g(A) has the root space decomposition $g(A)=\bigoplus_{\alpha\in Q} g_\alpha(A)$ where

$g_\alpha(A)=\{x\in g(A)/[h,x]=\alpha(h)x, \text{ for all } h\in H\}$.

An element α, $\alpha\neq 0$ in Q is called a root if $g_\alpha\neq 0$. For any $\alpha\in Q$ and $\alpha=\sum_{k=1}^{n}k_i\alpha_i$ define support of α, written as supp α, by supp $\alpha=\{i\in N/k_i\neq 0\}$.

Let $\Delta=(=\Delta(A))$ denote the set of all roots of g(A) and Δ_+ the set of all positive roots of g(A);

$\Delta_-=-\Delta_+$ and $\Delta=\Delta_+\cup\Delta_-$.

Definition 2: [4][25] To every GCM A is associated a Dynkin diagram S(A) defined as follows: S(A) has n vertices and vertices i and j are connected by max{$|a_{ij}|$, $|a_{ji}|$} number of lines if $a_{ij}.a_{ji}\leq 4$ and there ia an arrow pointing towards I if $|a_{ij}|>1$. If $a_{ij}.a_{ji}>4$, i and j are connected by a bold faced edge, equipped with the ordered pair ($|a_{ij}|$, $|a_{ji}|$) of integers.

Definition 3: [4] A GCM A is called symmetrizable if DA is symmetric for some diagonal matrix D = diag(q1,…,qn), with qi> 0 and qi's are rational numbers. $A=(a_{ij})_{i,j=1}^n$ is symmetrizable if and only if there exists an invariant, bilinear, symmetric, non-degenerate form on g(A).

Definition 4: [17] Let $A=(a_{ij})_{i,j=1}^n$, be an indecomposable GCM of indefinite type. We define the associated Dynkin diagram S(A) to be of Quasi Affine (QA) type if S(A) has a proper connected sub diagram of affine type with n-1 vertices. The GCM A is of QA type if S(A) is of QA type. We then say the Kac-Moody algebra g(A) is of QA type.

Definition 5: [9] A root $\alpha\in\Delta$ is called real, if there exists a $w\in W$ such that w(α) is a simple root, and a root which is not real is called an imaginary root. An imaginary root α is called isotropic if (α,α) = 0.

$\alpha\in\Delta_+^{im}$ is called a minimal imaginary root (MI root, for short) if α is minimal in Δ_+^{im} with respect to the partial order on H*. By the symmetry of the root system, it is enough to prove the results for positive imaginary roots only.

Definition 6: [3] A root $\gamma\in\Delta^{im}$ is said to be strictly imaginary if for every $\alpha\in\Delta^{re}$, $\alpha+\gamma$ or $\alpha-\gamma$ is a root. Set of all strictly imaginary roots is denoted by Δ^{sim}.

Definition 7: [15] A root $\alpha\in\Delta_+^{im}$ is called purely imaginary if for any $\beta\in\Delta_+^{im}$, $\alpha+\beta\in\Delta_+^{im}$. The Kac – Moody algebra is said to have the purely imaginary property if every imaginary root is purely imaginary.

III. Study on the Real and Imaginary Roots of Particular Class of $QAG_2^{(1)}$

Throughout this study we consider only indecomposable, symmetrizable GCM. In this section, we consider a particular family of $QAG_2^{(1)}$ and determine short and long real roots. Some of the imaginary roots are identified, minimal imaginary and isotropic roots up to height 3 among these imaginary roots are then listed. We also show that this family satisfies the purely imaginary property.

Proposition 1: The indefinite quasi affine Kac Moody algebras $QAG_2^{(1)}$ satisfy purely imaginary root. i.e. all the imaginary roots are purely imaginary.

Proof : The characterization of Kac Moody algebras with the purely imaginary property is given in [10], which states that any indecomposable symmetrizable GCM of order up to order 4 satisfies the purely imaginary property. The GCM associated with $QAG_2^{(1)}$ is indecomposable and since we restrict our study to symmetrizable GCM, it follows that all imaginary roots are purely imaginary in $QAG_2^{(1)}$.

Example 1 : Let the GCM A= $\begin{pmatrix} 2 & -1 & 0 & -a \\ -1 & 2 & -1 & -b \\ 0 & -3 & 2 & -3c \\ -a & -b & -c & 2 \end{pmatrix}$,

A is symmetrizable, A=DB

$$=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 & -a \\ -1 & 2 & -1 & -b \\ 0 & -1 & 2/3 & -c \\ -a & -b & -c & 2 \end{pmatrix};$$

We compute the non generate symmetric bilinear form (,) as follows:

$(\alpha_1,\alpha_1)=2$, $(\alpha_1,\alpha_2)=-1$, $(\alpha_1,\alpha_4)=-a$, $(\alpha_2,\alpha_2)=2$, $(\alpha_2,\alpha_3)=-1$, $(\alpha_3,\alpha_3)=2/3$, $(\alpha_2,\alpha_4)=-b$, $(\alpha_3,\alpha_4)=-c$, $(\alpha_4,\alpha_4)=2$

Length of real roots : $|\alpha_i|^2=2, 2, 2/3, 2$ for i=1,2,3,4 respectively.

Short real root is α_3;

Long real roots are α_1, α_2, α_4;

Roots of height 2:

$(\alpha_1+\alpha_2, \alpha_1+\alpha_2)>0$, $\alpha_1+\alpha_2$ is a real root.

$(\alpha_2+\alpha_3, \alpha_2+\alpha_3)>0$, $\alpha_2+\alpha_3$ is a real root.

$(\alpha_1+\alpha_4, \alpha_1+\alpha_4)=4-2a$

$\alpha_1+\alpha_4$ is imaginary iff $4-2a\leq 0$

Also $\alpha_1+\alpha_4$ is minimal imaginary iff $a\geq 2$

$\alpha_1+\alpha_4$ is isotrophic iff a=2

Consider $(\alpha_2+\alpha_4, \alpha_2+\alpha_4)=4-2b$;

$\alpha_2+\alpha_4$ is minimal imaginary iff $b\geq 2$

$\alpha_3+\alpha_4$ is isotrophic iff b=2

Now, $(\alpha_3+\alpha_4, \alpha_3+\alpha_4) = (8/3)-2c$
$\alpha_3+\alpha_4$ is minimal imaginary iff $3c \geq 4$
$\alpha_3+\alpha_4$ is isotrophic iff $4/3 = c$

Roots of height 3:

$(\alpha_1+ \alpha_2+ \alpha_3, \alpha_1+ \alpha_2+ \alpha_3) >0$;
$(\alpha_1+ \alpha_2+ \alpha_4, \alpha_1+ \alpha_2+ \alpha_4)= 6-2(1+a+b)$
$\alpha_1+ \alpha_2+ \alpha_4$ is isotropic iff $3=1+a+b$
$\alpha_1+ \alpha_2+ \alpha_4$ is imaginary iff $3 \leq 1+a+b$
$(\alpha_2+ \alpha_3+ \alpha_4, \alpha_2+ \alpha_3+ \alpha_4)= (14/3)-2(1+a+b)$
$\alpha_2+ \alpha_3+ \alpha_4$ is imaginary iff $7 \leq 3(1+a+b)$
$\alpha_2+ \alpha_3+ \alpha_4$ is isotropic iff $7=3(1+a+b)$
$(\alpha_1+2\alpha_2, \alpha_1+2\alpha_2) > 0$
$(\alpha_1+2\alpha_4, \alpha_1+2\alpha_4)=6-2a$
$\alpha_1+2\alpha_4$ is isotropic iff $a=3$
$\alpha_1+2\alpha_4$ is imaginary iff $3 \leq a$.

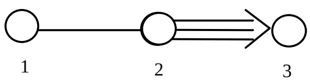
Now $(2\alpha_1+\alpha_2, 2\alpha_1+\alpha_2) > 0$;
$(2\alpha_1+\alpha_4, 2\alpha_1+\alpha_4)=6-2a$; $2\alpha_1+\alpha_4$ is isotropic iff $a=3$
$2\alpha_1+\alpha_4$ is imaginary iff $3 \leq a$
$(\alpha_2+2\alpha_3, \alpha_2+2\alpha_3) < 0$, $\alpha_2+2\alpha_3$ is a imaginary root
$(\alpha_2+2\alpha_4, \alpha_2+2\alpha_4)=6-4b$; $\alpha_2+2\alpha_4$ is imaginary iff $3 \leq 2b$
$(2\alpha_2+\alpha_3, 2\alpha_2+2\alpha_3) > 0$;
$(2\alpha_2+\alpha_4, 2\alpha_2+2\alpha_4)=6-4b$;
$2\alpha_2+\alpha_4$ is imaginary iff $3 \leq 2b$
$(\alpha_3+2\alpha_4, \alpha_3+2\alpha_4)=(14/3)-4c$
$\alpha_3+2\alpha_4$ is imaginary iff $7 \leq 6c$;
$(2\alpha_3+\alpha_4, 2\alpha_3+\alpha_4)=(10/3)-4c$
$2\alpha_3+\alpha_4$ is imaginary iff $5 \leq 6c$

IV. Complete Classification of Dynkin Diagrams Associated with $QAG_2^{(1)}$

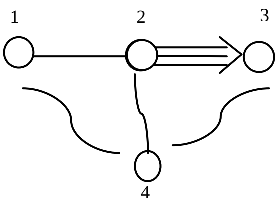
In this section we prove the classification theorem, in a more general case; we prove that there are 999 non-isomorphic, connected Dynkin diagrams associated with the indefinite Quasi affine Kac-Moody algebras $QAG_2^{(1)}$. Also we identify number of extended hyperbolic and indefinite non-extended hyperbolic diagrams in this class.

Theorem 1 (Classification theorem) : In general, there are 999 non isomorphic, connected Dynkin diagrams associated with the indefinite Quasi affine Kac-Moody algebras $QAG_2^{(1)}$.

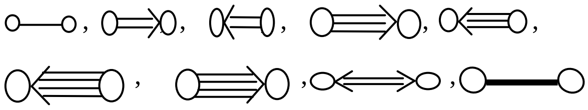
Proof. Consider the affine Kac-Moody algebra $G_2^{(1)}$ with 3 vertices whose Dynkin diagram is given by



Add the fourth vertex, which is connected to at least one of the three vertices of $G_2^{(1)}$.

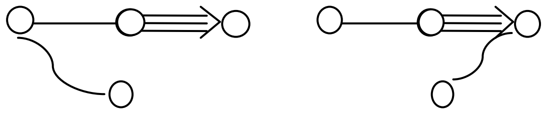


Here  denotes the 9 possible connections of vertex 4 with the other 3 vertices.



The fourth vertex can be connected to the existing three vertices by one the following cases:

Case i) Fourth vertex is connected to exactly one of the three vertices1,2,3. In this case, fourth vertex can be connected to the i^{th} vertex, (i=1,2,3) by the 9 possible edges given above. This single vertex 'i' can be chosen in 3 ways.



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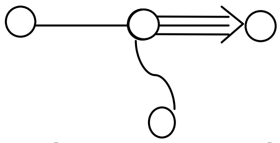
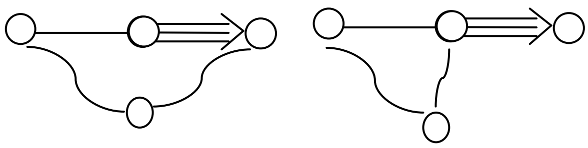


Hence, the maximum number of connected Dynkin diagrams associated with $QAG_2^{(1)}$ will be $3C_1 \times 9 = 27$

Case ii) Fourth vertex is connected to exactly two of the 3 vertices; those two vertices can be chosen from the three vertices in $3C_2$ possible ways. (i.e.(1,4) and (2,4) can be joined or (1,4) and (3,4) can be joined or (2,4) and (3,4) can be joined.)



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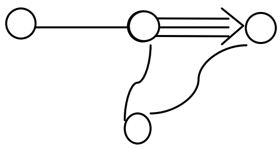
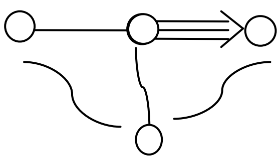


For each possibility, the remaining edge can be connected by any of the above mentioned 9 edges.

Hence. in this case, the associated connected Dynkin diagrams are $3C_2 x\, 9^2 = 243$

Case iii) Fourth vertex is connected to all the 3 vertices. In this case, there are 9^3 connected Dynkin diagrams.



Totally there are 27 + 243 + 729 =999 non isomorphic, connected Dynkin diagrams associated with $QAG_2^{(1)}$.

The 729 Dynkin diagrams covered under case iii) has already been given in [18]. Hence we list below the remaining 270 diagrams covering case i) and case ii).

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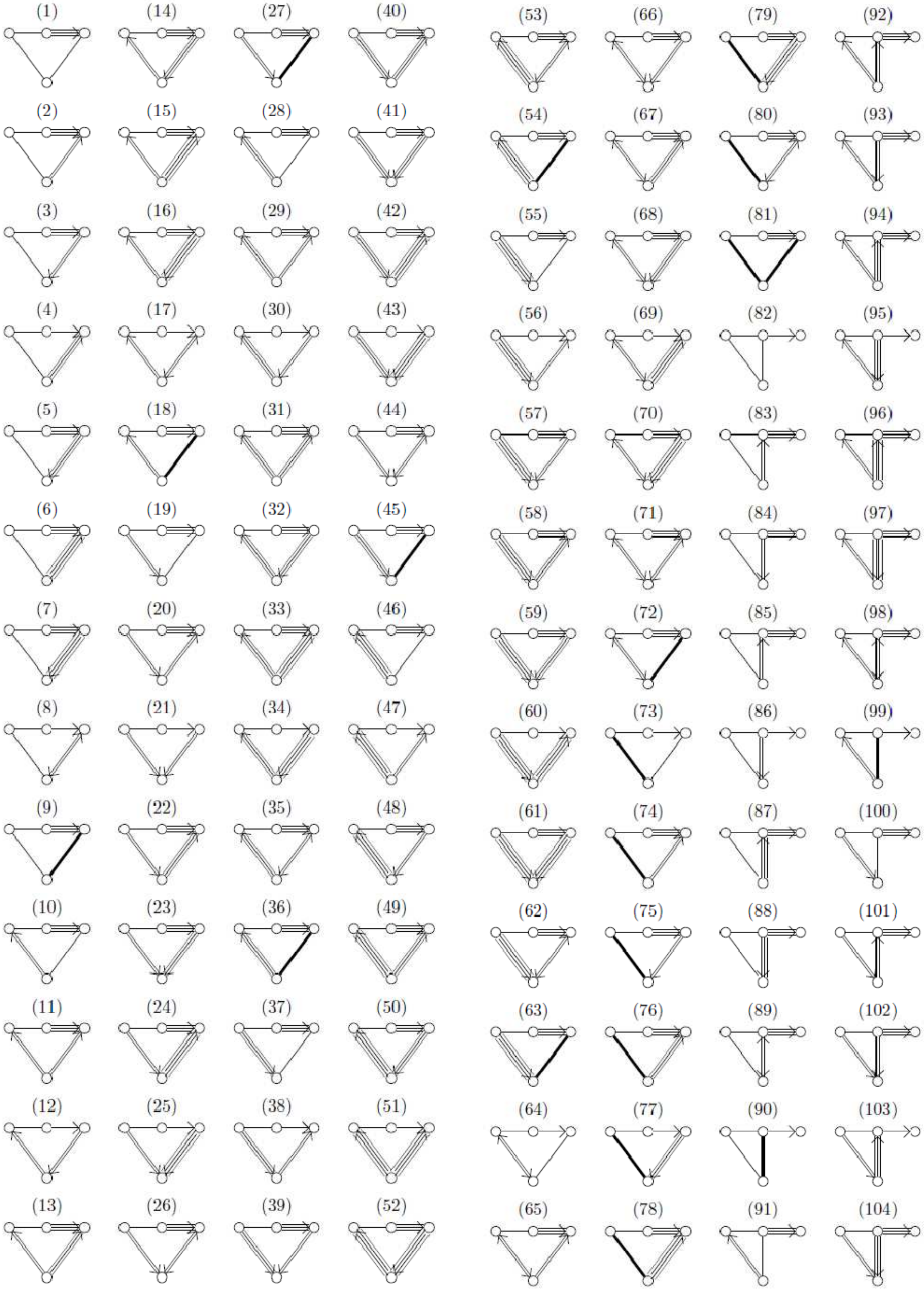

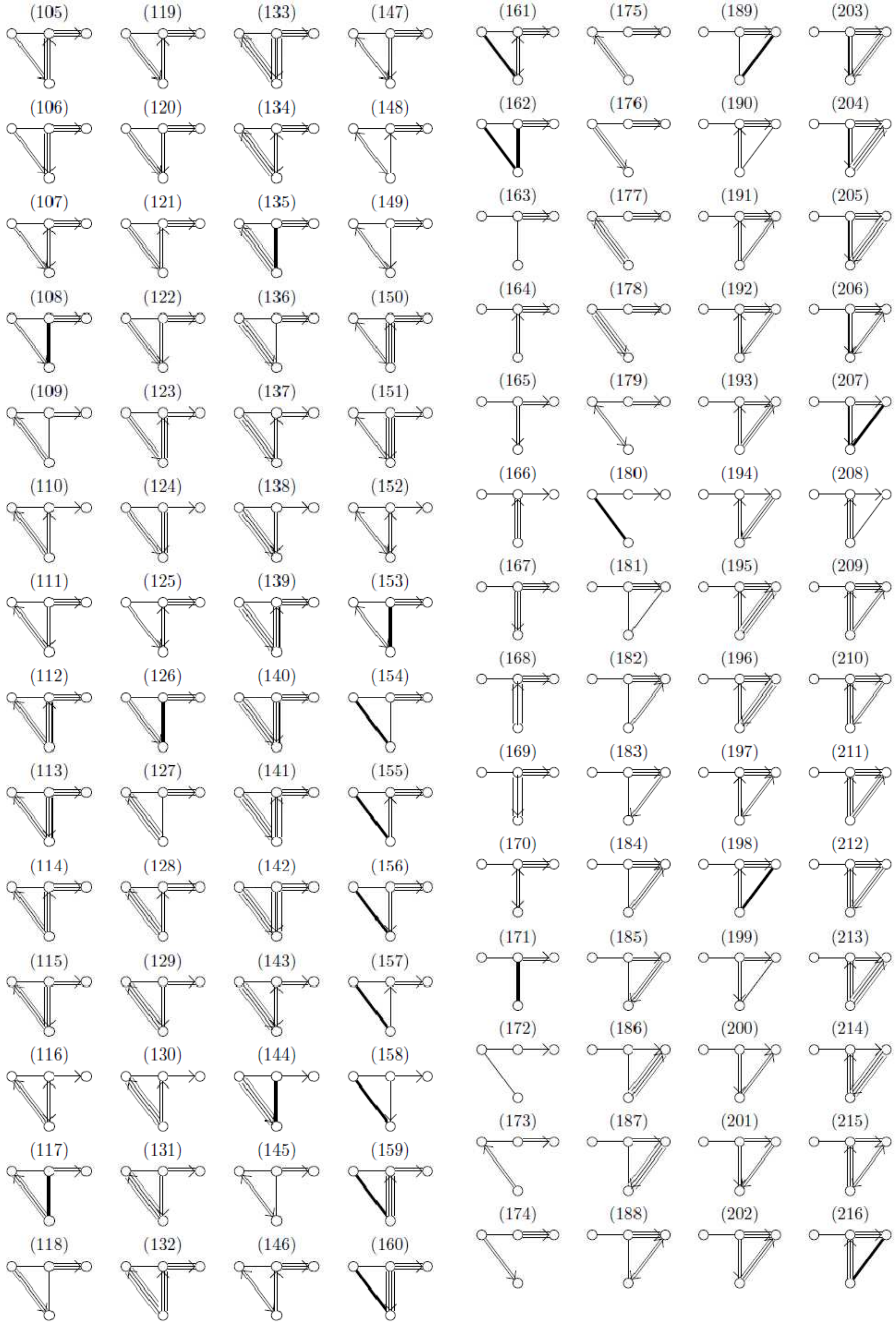
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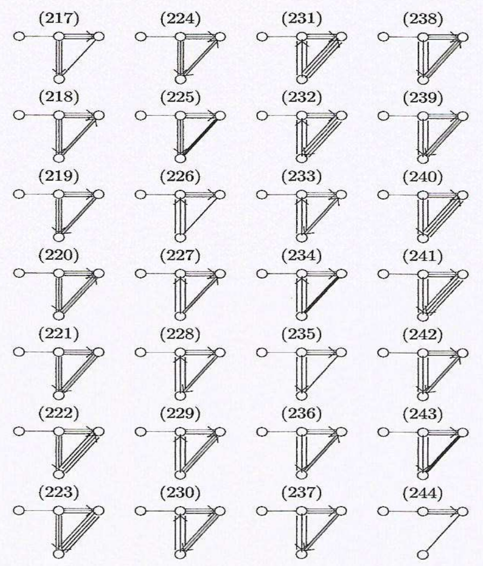

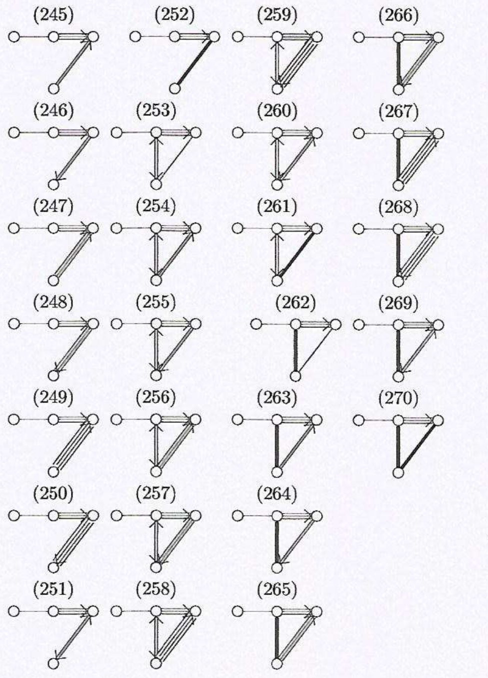

Proposition 2 : Among the 999 Dynkin diagrams, 728 families are extended hyperbolic and 271 Dynkin diagrams in $QAG_2^{(1)}$ are indefinite, which are neither hyperbolic nor extended hyperbolic.

Proof : By definition, those Dynkin diagrams that contain a bold faced edge can not be of extended hyperbolic type; 271 Dynkin diagrams contain bold faced edges and hence they are neither hyperbolic nor of extended hyperbolic type. Still they are indefinite type of Dynkin diagrams.

All the other Dynkin diagrams which do not contain a bold faced edge are of extended hyperbolic, since every proper connected sub diagram is either finite or affine . Hence, the remaining 728 Dynkin diagrams are all of extended hyperbolic type, since each proper connected sub diagrams in these cases are of finite, affine or hyperbolic types.

V. Conclusion

In this paper, complete classification of Dynkin diagrams in the general class of $QAG_2^{(1)}$ and some properties of real and imaginary roots are obtained for the family $QAG_2^{(1)}$. The study can be extended further to understand the structure and using the representation theory, the multiplicities of roots can be computed.

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