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Some Constructions of Affine Resolvable Designs with Unequal Block Sizes

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Abstract **– Some construction methods of affine resolvable balanced incomplete block designs and affine resolvable rectangular type partially balanced incomplete block designs with unequal block sizes are proposed with illustrations, which are based on the incidence matrices of the known affine resolvable balanced incomplete block designs.**

Keywords **– Affine Resolvable Design, Balanced Incomplete Block Design, Efficiency Balanced Design, Partially Balanced Incomplete Block Design, Rectangular Design, Resolvable Design, Variance Balanced Design.**

I. Introduction

The concept of resolvability and affine resolvability was introduced by Bose [1] in the year 1942. A block design is said to be resolvable if the b blocks each of size k can be grouped into r resolution sets of b/r blocks each such that in each resolution set every treatment is replicated exactly once. Bose [1] proved that necessary condition for the resolvability of a BIBD is $b \geq v + r - 1$. A resolvable block design is said to be affine resolvable if and only if $b = v + r - 1$ and any two blocks belonging to different resolution sets intersect in the same number, say, $q_2 = k^2/v$ of treatments.

The concept of resolvability and affine resolvability was generalized by Shirkhande and Raghavarao [2] to μ - resolvability and affine μ- resolvability. An incomplete block design with parameters $v, b = \beta t, r = \mu t, k$ is said to be μ-resolvable if the b blocks can be divided into t sets of β each, such that each treatment occurs μ times in each set of blocks. Further, μ-resolvable incomplete block design is said to be affine μ-resolvable if every two distinct blocks from the same μ - resolution set intersect in the same number, say, q_1, of treatments, whereas every two blocks belonging to different μ-resolution sets intersect in the same number, say, q_2, of treatments. Necessary and sufficient condition for the μ - resolvable BIB design to be affine μ - resolvable with the block intersection numbers q_1 and q_2 is $q_1 = k(\mu - 1)/(\beta - 1)$ $= k + \lambda - r$ and $q_2 = \mu k/\beta = k^2/v$. After this there has been a very rapid development in the area of experimental designs. Prominent work has been done by Bailey et al. [3], Caliński et al. [5]-[8], Kageyama [9]-[12], Kageyama et al. [13], [14] and many others in this area of research; see [15]-[18].

The concept of μ - resolvability was further generalized to $(\mu_1, \mu_2, \ldots, \mu_t)$ resolvability by Kageyama [11] in 1976. A block design is said to be $(\mu_1, \mu_2, \ldots, \mu_t)$ – resolvable if the blocks can be separated into t sets of mi (≥ 2) blocks such that the set consisting of m_i blocks contains every treatment exactly $\mu_i(\geq 1)$ times, i.e. the set of m_i blocks form a μ_i- replication set of each treatment ($i = 1, 2, \ldots t$). Furthermore, when $\mu_1 = \ldots = \mu_t$ $(= \mu$, say), it is simply called μ - resolvable for $\mu \geq 1$. We consider only those $(\mu_1, \mu_2, \ldots, \mu_t)$–resolvable block designs which have constant block size within each set. The constant block size within the l^{th} set is denoted by k_l^* for $l = 1,2,\ldots,t$. A $(\mu_1, \mu_2, \ldots, \mu_t)$-resolvable block design with a constant block size within each set will be said to be affine $(\mu_1, \mu_2, \ldots, \mu_t)$ - resolvable if:

(i) For l = 1,….,t every two distinct block from the l^{th} set intersect in the same number say q_{ll} of treatments.

(ii) For $l \neq l'$ =1,…..,t every block from the l^{th} set intersects every block of the l'^{th} set in the same number, say $q_{ll'}$ of treatments.

It is evident that for affine $(\mu_1, \mu_2, \ldots, \mu_t)$ – resolvable designs

$$q_{ll}(m_l - 1) = k_l^*(\mu_l - 1) \text{ and } q_{ll'} m_{l'} = k_l^* \mu_{l'}.$$

In this paper we have proposed construction methods of affine resolvable rectangular type partially balanced incomplete block designs with unequal block sizes by using incidence matrix of an affine resolvable balanced incomplete block design. Kageyama [19] gave the construction method of affine resolvable designs with unequal block sizes which were based on the incidence matrices of the known affine resolvable balanced incomplete block design. He proved that these designs are variance balanced. In this paper we proposed that these constructed designs are efficiency balanced as well.

Let us consider v treatments arranged in b blocks, such that the j^{th} block contains k_j experimental units and i^{th} treatment appears r_i times in the entire design, i = 1, 2, ..., v; j = 1, 2, …, b. For any block design there exist a incidence matrix $N = [n_{ij}]$ of order $v \times b$, where n_{ij} denotes the number of experimental units in the j^{th} block getting the i^{th} treatment. When n_{ij} = 1 or 0 ∀ i and j, the design is said to be binary. Otherwise it is said to be nonbinary. In this paper we consider binary block designs only. The following additional notations are used $\underline{k} = [k_1, k_2, \ldots.. k_b]'$ is the column vector of block sizes, $\underline{r} = [r_1, r_2, \ldots\ldots, r_v]'$ is the column vector of treatment

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replication, $K_{b\times b}$ = diag $[k_1, k_2, ...k_b]$, $R_{v\times v}$ = diag $[r_1, r_2, ..., r_v]$, $\sum r_i = \sum k_j = n$ is the total number of experimental units, with this $N1_b = \underline{r}$ and $N'1_v = \underline{k}$, where 1_a is the $a\times 1$ vector of ones.

An equi-replicate, equi-block sized, incomplete design, which is also balance in the sense given above is called balanced incomplete block design, which is an arrangement of v symbols (treatment) into b sets (blocks) each containing $k(< v)$ distinct symbols, such that any pair of distinct symbols occurs in exactly λ sets. Then it is easy to see that treatments occur in $r(> \lambda)$ sets. v, b, r, k, λ are called parameters of the BIBD and the parameters satisfies the relations $vr = bk$, $r(k-1) = \lambda(v-1)$ and $b \geq v$ (Fisher's inequality).

A partially balanced incomplete block design based on an m-association scheme, with parameters v, b, r, k, λ_i $(i = 1,2,...., m)$, is a block design with v treatments and b blocks of size k each such that every treatment occurs in r blocks and two distinct treatments being i^{th} associate occur together in exactly λ_i blocks.

Rectangular designs, introduced by Vartak [20], are 3-associate PBIB designs based on a rectangular association scheme of mn treatments arranged in an $m\times n$ rectangular array such that, with respect to each treatment, the first associate are the other $m-1(=n_1$, say) treatments of the same column, the second associates are the other $n-1(=n_2$, say) treatments of the same row and the remaining $(m-1)(n-1)(=n_3$, say) treatments are the third associates. For the definitions of PBIB design and rectangular design along with their combinatorial properties, refer, Raghavarao [21]. Rectangular designs have been studied by Banerjee, Bhagwandas and Kageyama [22], Bhagwandas, Banerjee and Kageyama [23] Kageyama and Sinha [24], Banerjee and Kageyama [25] and many others.

Though there have been balanced designs in various senses (see [26], [6]). We will consider a balance design of the following types (i) Variance Balanced (ii) Efficiency Balanced. Out of these two main concepts of balancing, Rao [27] gives a necessary and sufficient condition for a general block design to be variance balanced. The concept of efficiency balanced was introduced by Jones [28] and the nomenclature "Efficiency Balanced" is due to Puri et al. [26] and Williams [29].The importance of variance-balance and resolvability in the context of experimental planning is well known; the former yields optimal designs apart from ensuring simplicity in the analysis and the latter is helpful among other respects, in the recovery of interblock information. Also practical situations sometimes demand designs with varying block sizes (see [30]) and affine resolvable design with unequal replication numbers between sets of blocks; for a practical example, see Kageyama [11].

Variance Balanced : A block design is called variance balanced if and only if it permits the estimation of all normalized treatment contrasts with the same variance.

Let us consider the matrix $C = R - NK^{-1}N'$

Where R = diag $(r_1, r_2, ...r_v)$, K = diag $(k_1, k_2, ...k_b)$

Kageyama [10] showed that N is variance balanced block design if and only if

$$C = \eta\left(I_v - \frac{1}{v}1_v 1_v'\right)$$

Where η is the unique non zero eigen value of C with multiplicity $(v-1)$, I_v is the $v\times v$ identity matrix, 1_v is $v\times 1$ vector all of whose elements are one.

Efficiency Balanced: A block design is called efficiency balanced if every contrast of treatment effects is estimated through the design with the same efficiency factor.

Let us consider the matrix M_0 given by Caliński [5]

$$M_0 = R^{-1}NK^{-1}N' - \frac{1}{n}1_v r' \quad (1)$$

and since $M_0S = \mu S$, where μ is the unique non zero eigen value of M_0 with multiplicity $(v-1)$ and M_0 is given as (1).

Caliński [5] showed that for such designs every treatment contrast is estimated with the same efficiency $(1-\mu)$ and N is an efficiency balanced design if and only if

$$M_0 = \mu\left(I_v - \frac{1}{n}1_v r'\right) \quad (2)$$

Kageyama [10] proved that for the efficiency balanced block design N, eq^n (2) is fulfilled if and only if

$$C = (1-\mu)\left(R - \frac{1}{n}rr'\right)$$

II. Construction of Design Matrix: Method I

Let N be the $v\times b$ incidence matrix of an affine resolvable balanced incomplete block design D with parameters $v = 2k, b = 4k-2, r = 2k-1, \lambda = k-1$ and N^c is incidence matrix of complement of the design D. Then the following incidence matrix N^* yields an affine resolvable rectangular type partially balanced incomplete block design D^* with unequal block sizes.

$$N^* = \begin{bmatrix} N & N & N^c & \underline{1}_v & \underline{0}_v & \underline{0}_v \\ N^c & N & N & \underline{0}_v & \underline{1}_v & \underline{0}_v \\ N & N^c & N & \underline{0}_v & \underline{0}_v & \underline{1}_v \end{bmatrix} \quad (3)$$

In incidence matrix N^*, $\underline{1}_v$ is the $v\times 1$ unit vector all of whose elements are unity and $\underline{0}_v$ is the $v\times 1$ zero vector all of whose elements are zero.

The rectangular association scheme having mn treatments (in the present construction method $m = 2k$ and $n = 3$) can be arranged in a rectangular array of m rows and n columns displayed in the following manner

$$\begin{array}{cccc} 1 & 2k+1 & \cdots & 4k+1 \\ 2 & 2k+2 & \cdots & 4k+2 \\ \vdots & \vdots & & \vdots \\ 2k & 4k & \cdots & 6k \end{array} \quad (4)$$

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Where $n_1 = m-1, n_2 = n-1$ and $n_3 = (m-1)(n-1)$. The resultant design is also affine resolvable as $b = v + r - 1$ and k^2/v is an integer.

Theorem 2.1 : The existence of an affine resolvable balanced incomplete block design with parameters $v = 2k, b = 4k-2, r = 2k-1, k, \lambda = k-1$ (where k is even) implies the existence of an affine resolvable rectangular type 3-associates partially balanced incomplete block design with parameters $v^* = 6k,\ b^* = 12k-3, r^* = 6k-2,\ k_1^* = 3k,\ k_2^* = 2k,\ \lambda_1^* = 3k-2,\ \lambda_2^* = 2k-1$ and $\lambda_3^* = 3k-1$; having a rectangular association scheme of $n_1 = 2k-1, n_2 = 2, n_3 = 2(2k-1)$.

Proof: Consider an affine resolvable balanced incomplete block design D with parameters $v = 2k, b = 4k-2, r = 2k-1, k, \lambda = k-1$ and N^c be the incidence matrix of the complement of the design D. Under the present method of construction, the design D^* yields the parameters $v^* = 6k,\ b^* = 12k-3,\ r^* = 6k-2,\ k_1^* = 3k$ and $k_2^* = 2k$, which are obvious. Among $6k$ treatments a rectangular association scheme can be naturally defined as follows. These $6k$ treatments are arranged in a rectangular array of $2k$ rows and 3 columns such that first associates of any treatment θ (say) are $(2k-1)$ treatments other than this treatment of the same column, the second associates are other 2 treatments in the same row and the remaining $2(2k-1)$ treatments are the third associates of θ. Further, the parameters λ_i^*(i=1, 2, 3) can be determined as follows: Let us number the rows of N^* as $1,2,\dots,2k, 2k+1,\dots,4k, 4k+1,\dots,6k$. In the present construction method any (θ,ϕ) pair, with the combinations of $\frac{N}{N^c}$ or $\frac{N}{N}$ occurs as given below

$$\begin{array}{l} N = \overbrace{\begin{pmatrix}1\\1\end{pmatrix}}^{k-1}\ \overbrace{\begin{pmatrix}1\\0\end{pmatrix}}^{k}\ \overbrace{\begin{pmatrix}0\\1\end{pmatrix}}^{k}\ \overbrace{\begin{pmatrix}0\\0\end{pmatrix}}^{k-1} \\ \hline N^c = \begin{pmatrix}0\\0\end{pmatrix}\ \begin{pmatrix}0\\1\end{pmatrix}\ \begin{pmatrix}1\\0\end{pmatrix}\ \begin{pmatrix}1\\1\end{pmatrix} \end{array} \qquad \text{(i)}$$

$$\begin{array}{l} N = \overbrace{\begin{pmatrix}1\\1\end{pmatrix}}^{k-1}\ \overbrace{\begin{pmatrix}1\\0\end{pmatrix}}^{k}\ \overbrace{\begin{pmatrix}0\\1\end{pmatrix}}^{k}\ \overbrace{\begin{pmatrix}0\\0\end{pmatrix}}^{k-1} \\ \hline N = \begin{pmatrix}1\\1\end{pmatrix}\ \begin{pmatrix}1\\0\end{pmatrix}\ \begin{pmatrix}0\\1\end{pmatrix}\ \begin{pmatrix}0\\0\end{pmatrix} \end{array} \qquad \text{(ii)}$$

(a) From structure given in (3), in the I[st] partition $\left[N \mid N \mid N^c \right]$, the inner product of any two rows of N or N^c is $(k-1)$. So the value of λ_1^* can be calculated as $\lambda_1^* = (k-1) + (k-1) + (k-1) + 1 = 3k-2$.

(b) From structure given in (i) and (ii), we can see that there is no contribution of first row of N with first row of N^c, but the first row of N is contributing with first row of N. So the value of λ_2^* can be calculated as $\lambda_2^* = 2k-1$.

(c) From structure given in (i) and (ii), we can see that the inner product of first row of N with the $(2,3,\dots,4k,\ 4k+2,\ \dots,6k)^{th}$ row of N^c is k and inner product of first row of N with $(2,3,\dots,4k,\ 4k+2,\ \dots,6k)^{th}$ row of N is $(k-1)$. So the value of λ_3^* can be calculated as $\lambda_3^* = (k-1) + k + k = 3k-1$.

Here, $6k$ treatments are arranged in a $2k \times 3$ rectangular array such that, with respect to each treatment, the first associate are the other $2k-1 (= n_1)$ treatments of the same column, the second associates are the other $2 (= n_2)$ treatments of the same row and the remaining $2(2k-1) (= n_3)$ treatments are the third associates. In the present method of construction the affine resolvability of the resultant design is easily shown as, $b^* = v^* + r^* - 1$, intersection within the resolution sets $q_{ll} = 0\ (l = 1,\dots,r^*)$ and intersection between the resolution sets $q_{ll'} = k_1^* k_2^* / v^* = k\ (l \neq l' = 1,\dots,r^*)$. This completes the proof.

Example 2.2 : Let us consider an affine resolvable balanced incomplete block design with parameters $v = 4, b = 6, r = 3, k = 2, \lambda = 1$ with incidence matrix N given through the blocks [(1,2), (3,4)], [(1,3), (2,4)], [(1,4), (2,3)] and N^c is the complement of incidence matrix N. Theorem 2.1 yields an affine resolvable rectangular type partially balanced incomplete block design with unequal block sizes and parameters $v^* = 12$, $b^* = 21,\ r^* = 10,\ k_1^* = 6,\ k_2^* = 4,\ \lambda_1^* = 4,\ \lambda_2^* = 3,\ \lambda_3^* = 5$.
The incidence matrix of the resultant design is given as follows:

$$N^* = \left[\begin{array}{cc:cc:cc:cc:cc:cc:cc:cc:cc:ccc}
1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&1&0&0\\
1&0&0&1&0&1&1&0&0&1&0&1&0&1&1&0&1&0&1&0&0\\
0&1&1&0&0&1&0&1&1&0&0&1&1&0&0&1&1&0&1&0&0\\
0&1&0&1&1&0&0&1&0&1&1&0&1&0&1&0&0&1&1&0&0\\
\hdashline
0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0\\
0&1&1&0&1&0&1&0&0&1&0&1&1&0&0&1&0&1&0&1&0\\
1&0&0&1&1&0&0&1&1&0&0&1&0&1&1&0&0&1&0&1&0\\
1&0&1&0&0&1&0&1&0&1&1&0&0&1&0&1&1&0&0&1&0\\
\hdashline
1&0&1&0&1&0&0&1&0&1&0&1&1&0&1&0&1&0&0&0&1\\
1&0&0&1&0&1&0&1&1&0&1&0&1&0&0&1&0&1&0&0&1\\
0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&0&0&1\\
0&1&0&1&1&0&1&0&1&0&0&1&0&1&0&1&1&0&0&0&1
\end{array}\right]$$

The rectangular association scheme is written as

1	5	9
2	6	10
3	7	11
4	8	12

In the above design
$b^* = 12 + 10 - 1 = 21,\ q_{ll} = 0 (l = 1,\dots.,9)$,
$q_{ll} = (k_1^*)^2 / v^* = 3 (l = 1,\dots,8)$ and $q_{ll'} = k_1^* k_2^* / v^* = 2$
$(l \neq l' = 1,\dots.,9)$. Hence the designs Constructed above also possess the property of affine resolvability.

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III. Construction of Design Matrix: Method II

Kageyama [19] presented a method of constructing affine resolvable design with unequal block sizes. According to kageyama [19], if N be the $v\times b$ incidence matrix of an affine resolvable balanced incomplete block design D with parameters $v=2k, b=4k-2, r=2k-1, k$, $\lambda=k-1$ and N^c is incidence matrix of complement of the design D. Then the following incidence matrix N_* yields an affine resolvable variance balanced design D_* with unequal block sizes and $\eta_*=2k$.

$$N_*=\left[\begin{array}{c:c:c:c} I_v & \underline{1}_v & \underline{0}_v & N \\ \hdashline I_v & \underline{0}_v & \underline{1}_v & N^c \end{array}\right] \qquad (5)$$

In incidence matrix N_*, I_v is an $v\times v$ identity matrix, $\underline{1}_v$ is the $v\times 1$ unit vector all of whose elements are unity and $\underline{0}_v$ is the $v\times 1$ zero vector all of whose elements are zero. For the resolvability in design D_*, the first resolution set is from $\begin{pmatrix} I_v \\ I_v \end{pmatrix}$, second resolution set is from $\begin{pmatrix} \underline{1}_v & \underline{0}_v \\ \underline{0}_v & \underline{1}_v \end{pmatrix}$ and the remaining resolution sets can be formed by the original partition in the affine resolvable design D. Furthermore, the affine resolvability of the resultant design is easily shown, because of $b=v+r-1$.Here we proved that the resultant design given in (5) is also efficiency balanced.

Theorem 3.1. The existence of an affine resolvable balanced incomplete block design D with parameter $v=2k,\ b=4k-2,\ r=2k-1,\ k,\ \lambda=k-1$ implies the existence of an affine resolvable efficiency balanced block design with unequal block sizes and parameter $v_*=4k$, $b_*=6k$, $r_*=2k+1$, $k_{1*}=2$, $k_{2*}=2k$, $\lambda_*=k, q_{ll}=0(l=1,\dots,r_*), q_{1l}=1(1,\dots,r_*), q_{2l}=k(l=3,\dots,r_*)$, $q_{ll'}=k$ $(l\neq l'=3,\dots,r_*)$ and $\mu_*=1/(2k+1)$.

Proof: Let N be the $v\times b$ incidence matrix of an affine resolvable BIB design D and N^c be the incidence matrix of complement of the design D. In the present method of construction, the design D_* yields the parameters $v_*=4k$, $b_*=6k$, $r_*=2k+1$, $k_{1*}=2, k_{2*}=2k$; which are obvious. In the affine resolvable balanced incomplete block design D; any (θ,ϕ) pair occurs in λ blocks. Thus in the construction method given in (5), the frequency of (θ,ϕ) pair can be calculated as $\lambda_*=k-1+1=k$. Here b_* blocks are separated into r_* resolution sets, the first resolution set contains v blocks and the remaining resolution sets have only two blocks in each. Furthermore, the affine resolvability of the resultant design is easily shown as, $b_*=v_*+r_*-1=6k$, intersection within the resolution sets $q_{ll}=0(l=1,\dots,r_*)$, intersection between first and other resolution sets $q_{1l}=1(l=1,\dots,r_*)$ and intersection between the resolution sets $q_{ll'}=k$ $(l\neq l'=1,2,\dots,r_*)$. The matrix C of the design given in (5) is

$$C=RI_v-NK^{-1}N'$$

The calculation of the efficiency can be done as follows

$$NN'=(2k+1)I_v+k(J_{vv}-I_v)$$

$$NK^{-1}N'=\left(\frac{1}{2}+\frac{1}{2k}+\frac{2k-1}{2k}\right)I_v+\left(\frac{1}{2k}+\frac{k-1}{2k}\right)(J_{vv}-I_v)$$

$$R^{-1}NK^{-1}N'=\left(\frac{1}{2}+\frac{1}{2k}+\frac{2k-1}{2k}\right)\frac{1}{(2k+1)}I_v+\left(\frac{1}{2k}+\frac{k-1}{2k}\right)\frac{1}{(2k+1)}(J_{vv}-I_v)$$

$$R^{-1}NK^{-1}N'-\frac{1}{n}1_v r'=\left[\left(\frac{1}{2}+\frac{1}{2k}+\frac{2k-1}{2k}\right)\frac{1}{(2k+1)}I_v+\left(\frac{1}{2k}+\frac{k-1}{2k}\right)\frac{1}{(2k+1)}(J_{vv}-I_v)\right]-\frac{1}{4k(2k+1)}(2k+1)J_{vv}$$

$$R^{-1}NK^{-1}N'-\frac{1}{n}1_v r'=\left[\left(\frac{1}{2}+\frac{1}{2k}+\frac{2k-1}{2k}\right)\frac{1}{(2k+1)}I_v+\left(\frac{1}{2k}+\frac{k-1}{2k}\right)\frac{1}{(2k+1)}(J_{vv}-I_v)\right]-\frac{1}{4k}J_{vv}$$

$$M_0=\frac{1}{2k+1}[I_v-\frac{1}{4k}J_{vv}]$$

which yields $\mu^*=1/(2k+1)$. This completes the proof.

Example 3.2: Let us consider an affine resolvable balanced incomplete block design with parameters $v=8, b=14, r=7, k=4, \lambda=3$ with incidence matrix N given through the blocks [(1,2,4,7), (0,3,5,6)], [(2,3,5,7), (1,4,6,0)], [(3,4,6,7), (2,5,0,1)], [(4,5,0,7), (3,6,1,2)], [(5,6,1,7), (4,0,2,3)], [(6,0,2,7), (5,1,3,4)], [(0,1,3,7), (6,2,4,5)] and N^c is the complement of incidence matrix N. Theorem 3.1 yields affine resolvable variance and efficiency balanced design with unequal block sizes and parameters $v_*=16, b_*=24, r_*=9, k_{1*}=2, k_{2*}=8, \lambda_*=4$.

The incidence matrix of the resultant matrix is given as follows:

$$N^*=\left[\begin{array}{cccccccc:cc:cc:cc:cc:cc:cc:cc:cc}
1&0&0&0&0&0&0&0&1&0&0&1&0&1&0&1&1&0&0&1&1&0&1&0\\
0&1&0&0&0&0&0&0&1&0&1&0&0&1&0&1&0&1&1&0&0&1&1&0\\
0&0&1&0&0&0&0&0&1&0&1&0&1&0&0&1&0&1&0&1&1&0&0&1\\
0&0&0&1&0&0&0&0&1&0&0&1&1&0&1&0&0&1&0&1&0&1&1&0\\
0&0&0&0&1&0&0&0&1&0&1&0&0&1&1&0&1&0&0&1&0&1&0&1\\
0&0&0&0&0&1&0&0&1&0&0&1&1&0&0&1&1&0&1&0&0&1&0&1\\
0&0&0&0&0&0&1&0&1&0&0&1&0&1&1&0&0&1&1&0&1&0&0&1\\
0&0&0&0&0&0&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
\hdashline
1&0&0&0&0&0&0&0&0&1&1&0&1&0&1&0&0&1&1&0&0&1&0&1\\
0&1&0&0&0&0&0&0&0&1&0&1&1&0&1&0&1&0&0&1&1&0&0&1\\
0&0&1&0&0&0&0&0&0&1&0&1&0&1&1&0&1&0&1&0&0&1&1&0\\
0&0&0&1&0&0&0&0&0&1&1&0&0&1&0&1&1&0&1&0&1&0&0&1\\
0&0&0&0&1&0&0&0&0&1&0&1&1&0&0&1&0&1&1&0&1&0&1&0\\
0&0&0&0&0&1&0&0&0&1&1&0&0&1&1&0&0&1&0&1&1&0&1&0\\
0&0&0&0&0&0&1&0&0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0\\
0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1
\end{array}\right]$$

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The above structure is variance as well as efficiency balanced and the matrix C is given as $C = 8[I_{16} - (1/16)1_v 1_v^{'}]$ also $\eta_* = 8$ and $\mu_* = 0.1111$

IV. Results and Discussion

The following tables provide a list of affine resolvable designs with unequal block sizes for r ≤ 25; which can be obtained by using certain known affine resolvable balanced incomplete block designs.

Table 1. For Method I

S. No.	v^*	b^*	r^*	k_1^*	k_2^*	λ_1^*	λ_2^*	λ_3^*	m	n	Ref. No.**
1.	12	21	10	6	4	4	3	5	4	3	R(1)
2.	24	45	22	12	8	8	7	11	8	3	R(15)

Table 2. For Method II

S. No.	v_*	b_*	r_*	k_{1*}	k_{2*}	λ_*	η_*	μ_*	Ref. No.**
1	8	12	5	2	4	2	4	0.2	R(1)
2	16	24	9	2	8	4	8	0.1111	R(15)
3	24	36	13	2	12	6	12	0.769	R(36),MH(49)
4	32	48	17	2	16	8	16	0.0588	R(53),MH(97)
5	40	60	21	2	20	10	20	0.0476	MH(197)
6	48	72	25	2	24	12	24	0.04	R*(217)

**The symbols R (α), MH (α) and R* (α) denote the reference number α in Raghavarao [21], Marshal Halls [31] and Rao [32] list.

V. Conclusion

The results given in this paper produce affine resolvable rectangular type partially balanced incomplete block designs with unequal block sizes. In this research paper we have also shown that the affine resolvable designs with unequal block sizes are variance as well as efficiency balanced. Further there is a scope to propose different methods of construction to obtain affine resolvable designs, which are efficiency as well as variance balanced.

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