

# Designing of a Controller by Using Model Order Reduction Techniques

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**Abstract** – Modeling of physical systems usually results in system of higher order generally greater than two. Design of controllers for such physical system becomes tedious when system order is high. So it is desirable to approximate these models by reduced order models. In this paper a reduced order model is obtain by pole clustering and factor division algorithm. The controller is designed for the reduced order model and is connected in cascade with the original system to obtain the desired specifications. The proposed method assures the stability of the system under the reduced order model case. The proposed method is illustrated by a numerical example.

**Keywords** – Pole Cluster, Order Reduction, Factor Division, Stability, Transfer Function, PID Controller.

## I. INTRODUCTION

The quality of a reduced order model is judged by the way it is utilized, and the degree of its success in representing the desired characteristics of the system. One of the main objectives of order reduction is to design a controller of low order which can effectively control the original system high order system so that the overall system is of low order and is easy to understand. It is thus important that the model order reduction methods should reduce the high order controller to a low order controller without incurring too much error. Model reduction is based on open loop considerations while closed loop stability performance is of main concern in controller design. Pade approximation [1,2] is the method of model order reduction of the higher order system. This gives the simplification of a model after converting it into a reduced order model.

A different approach can be used to simplify a model which results are stable model. In this approach the numerator coefficients can be obtained by factor division method [3,4] and denominator coefficients can be obtained by pole clustering [5,6]. Several methods [7, 8, 9, 10 and 11] have been developed for designing a PID controller. In this paper a simple algebraic scheme is proposed to design a PID controller for Linear Time Invariant Continuous System. The closed loop transfer function of the reduced order models with PID controller are compared with the reference model transfer function in frequency domain.

## II. STATEMENT OF THE PROBLEM

### 2.1. PID Controller Transfer Function

PID controller can be mathematically represented as[12],

$$u(t) = k_1[e(t) + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{de(t)}{dt}] \quad (1)$$

Where  $u(t)$  and  $e(t)$  denotes the control and error signals of the system.  $k_1$  is the proportion gain, and represents the integral and derivative time constants respectively. The corresponding PID controller transfer function is given as

$$G_c(s) = k_1 [1 + \frac{1}{T_i s} + T_d s] \quad (2)$$

Equation (2) can be rewritten as

$$G_c(s) = k_1 + \frac{k_2}{s} + k_3 \quad (3)$$

$k_1$ ,  $k_2$  and  $k_3$  are represents the proportional, integral and derivative gain values of the controller

### 2.2. Higher Order Transfer Function

Let higher order system or process whose performance is unsatisfactory may be described by the transfer function

$$G(s) = \frac{N(s)}{D(s)} = \frac{A_{21} + A_{22}s + A_{23}s^2 + \dots + A_{2n}s^{n-1}}{A_{11} + A_{12}s + A_{13}s^2 + \dots + A_{1,n+1}s^n} \quad (4)$$

And a reference model having the desired performance is given.

### 2.3. Lower Order Transfer Function

To find a  $r$ th lower order model for the above continuous system, where  $r < n$  in the following form, such that the lower order model retains the characteristics of the original system and approximates its response as closely as possible for the same type of inputs.

$$R(s) = \frac{a_{21} + a_{22}s + a_{23}s^2 + \dots + a_{2r}s^{r-1}}{a_{11} + a_{12}s + a_{13}s^2 + \dots + a_{1,r+1}s^{r+1}} \quad (5)$$

Where,  $a_{2j}$  and  $a_{1i}$  are scalar constants.

Objective is to derive a controller such that the performance of the augmented process matches with that of the reference model. To reduce the computational complexities and difficulties of implementation, the higher order of the system is reduced into lower second order system. And PID controller is also derived for reduced order system.

## III. REDUCTION METHOD

The reduction procedure for getting the  $r$ <sup>th</sup>-order reduced models consists of the following two steps:

**Step-1:** Determination of the denominator polynomial for the  $k$ <sup>th</sup> -order reduced model using pole clustering techniques

The criterion for grouping the poles in one particular cluster is based on relative distance between the poles and desired order in the process of reduced order modeling. Since each cluster may be finally replaced by a single real pole, the following rules are used for clustering the poles to get the denominator polynomial for reduced order models.

- i. Separate cluster should be made for real poles and complex pole
- ii. Poles on the  $j\omega$  -axis have to be retained in the reduced order model.

The cluster centre can be formed using a simple method known as ‘inverse distance measure’, which is explained as follows:

Let,  $r$  real poles in one cluster be  $(p_i)$  then the Inverse Distance Measure (IDM) criterion identifies the cluster centre as

$$p_c = \left\{ \left( \sum_{i=1}^r \left( \frac{1}{p_i} \right) \right) \div r \right\}^{-1} \quad (6)$$

Where  $p_c$  is cluster centre from  $r$  real poles of the original system.

Then denominator polynomial for order reduced model can be obtained as

$$D_k(s) = (s - p_{c1})(s - p_{c2}) \dots \dots (s - p_{ck}) \quad (7)$$

Where  $p_{c1}, p_{c2}, \dots, p_{ck}$  are 1<sup>st</sup>, 2<sup>nd</sup>, ..., k<sup>th</sup> cluster centre

Step-2:

Determination of the numerator of  $k^{\text{th}}$  order reduced model using Factor Division algorithm [4]

After obtaining the reduced denominator, the numerator of the reduced model is determined as follows

$$N_k(s) = \frac{N(s)}{D(s)} \times D_k(s) = \frac{N(s)}{D_k(s)} \quad (8)$$

Where  $D_k(s)$  is reduced order denominator

There are two approaches for determining of numerator of reduced order model.

- (i) By performing the product of  $N(s)$  and  $D_k(s)$  as the first row of factor division algorithm and  $D(s)$  as the second row up to  $s^{k-1}$  terms are needed in both rows.
- (ii) By expressing  $N(s)D_k(s)/D(s)$  as  $N(s)/[D(s)/D_k(s)]$  and using factor division algorithm twice; the first time to find the term up to  $s^{k-1}$  in the expansion of  $D(s)/D_k(s)$  (i.e. put  $D(s)$  in the first row and  $D_k(s)$  in the second row, using only terms up to  $s^{k-1}$ ), and second time with  $N(s)$  in the first row and the expansion  $[D(s)/D_k(s)]$  in the second row.

Therefore the numerator  $N_k(s)$  of the reduced order model ( $R_k(s)$ ) in eq.(4) will be the series expansion of

$$\frac{N(s)}{D_k(s)} = \frac{\sum_{i=0}^{k-1} c_i s^i}{\sum_{i=0}^k d_i s^i} \quad (9)$$

About  $s=0$  up to term of order  $s^{k-1}$ .

This is easily obtained by using the Routh recurrence formulae to generate the third, fifth, and seventh etc rows as,

$$\alpha_0 = \frac{g_0}{h_0} < \begin{matrix} g_0 & g_1 & g_2 & \dots & g_{k-1} \\ h_0 & h_1 & h_2 & \dots & h_{k-1} \end{matrix}$$

$$\alpha_1 = \frac{l_0}{h_0} < \begin{matrix} l_0 & l_1 & l_2 & \dots & l_{k-2} \\ h_0 & h_1 & h_2 & \dots & h_{k-2} \end{matrix}$$

$$\alpha_2 = \frac{m_0}{h_0} < \begin{matrix} m_0 & m_1 & m_2 & \dots & m_{k-3} \\ h_0 & h_1 & h_2 & \dots & h_{k-3} \end{matrix}$$

$$\alpha_{k-2} = \frac{p_0}{h_0} < \begin{matrix} p_0 & p_1 \\ h_0 & h_1 \end{matrix}$$

$$\alpha_{k-1} = \frac{q_0}{h_0} < \begin{matrix} q_0 \\ h_0 \end{matrix}$$

Where

$$l_i = g_{i+1} - \alpha_0 * h_{i+1}, i=0,1,2, \dots$$

$$m_i = l_{i+1} - \alpha_1 * h_{i+1}, i=0,1,2, \dots$$

$$p_0 = p_1 - \alpha_{k-2} h_1$$

Therefore, the numerator  $N_k(s)$  of eq.(2) is given by

$$N_k(s) = \sum_{i=0}^{k-1} \alpha_i s^i \quad (10)$$

#### IV. GENERAL ALGORITHM FOR DESIGNING THE PID CONTROLLER

Step 1. Construction of a reference model  $M(s)$  on the basis of specifications whose closed loop system must approximate to that of the original closed loop response. Let it be specified as:

$$M(s) = \frac{a_0 + a_1 s + \dots + a_m s^m}{b_0 + b_1 s + \dots + b_n s^n} \quad (11)$$

Step 2. Determine an equivalent open loop specification model  $\overline{M}(s)$

$$\overline{M}(s) = \frac{M(s)}{1 - M(s)} \quad (12)$$

Step 3. Specified the structure of the controller

Let the controller structure is given by

$$R_c(s) = \frac{p_0 + p_1 s + \dots + p_k s^k}{q_0 + q_1 s + \dots + q_1 s^l} \quad (13)$$

Step 4. For determining the unknown controller parameters, the response of the closed loop system is matched with reference model as

$$R_c(s)R_p(s) = \overline{M}(s)$$

$$R_c(s) = \frac{\overline{M}(s)}{R_p(s)} = \sum_{i=0}^{\infty} e_i s^i \quad (14)$$

Where  $e_i$  are the power series expansion coefficients about  $s=0$ .

Step 5 Now the unknown control parameters  $p_i$  and  $q_i$  are obtained by equating the equation (13) and (14) in Pade sense

$$p_0 = q_0 e_0$$

$$p_1 = q_0 e_1 + q_1 e_0$$

$$p_2 = q_0 e_2 + q_1 e_1 + q_2 e_0$$

$$p_i = q_0 e_i + q_1 e_{i-1} + \dots + q_i e_0$$

$$0 = q_0 e_{i+1} + q_1 e_i + \dots + q_{i+1} e_0$$

$$0 = q_0 e_{i+j} + q_1 e_{i+j-1} + \dots + q_j e_0$$

The controller having the desired structure is obtained by solving above linear equations

Step 6 After obtaining the controller parameters, the close loop transfer function can be obtained as

$$R_{cl} = \frac{R_c(s)R_k(s)}{1+R_c(s)R_k(s)} \quad (15)$$

## V. NUMERICAL EXAMPLE

The proposed method explains by considering numerical example, taken from the literature.

Example:- Consider sixth order system  $G(s) = \frac{72+54s+12s^2+s^3}{100+180s+97s^2+18s^3+s^4}$

Step 1

Determination of Denominator of reduced order

Two cluster centers from the real poles -1,-2,-5,-10 can be formed equation since

$$P_{c1} = -1.01$$

$$P_{c2} = -2.003$$

$$D_2(s) = (s+1.01)(s+2.003)$$

$$D_2(s) = 2.023 + 3.013s + s^2$$

Step 2

Now using the factor division method the numerator of reduced order model is given as

Consider  $D_4(s)/D_2(s)$

$$\alpha_0 = 49.4 \begin{cases} 100 & 180 \\ 2.023 & 3.013 \end{cases}$$

$$\alpha_1 = 15.75 \begin{cases} 31.5 \\ 2 \end{cases}$$

Now Considering  $N_6/D_6(s)/D_2(s)$

$$\alpha_0 = 1.46 \begin{cases} 72 & 54 \\ 49.4 & 15.75 \end{cases}$$

$$\alpha_1 = 0.627 \begin{cases} 31.005 \\ 49.4 \end{cases}$$

Thus Reduced Numerator is given as

$$N_2(s) = 1.46 + 0.627s$$

Thus the Reduced model is given as

$$R_2(s) = \frac{1.46 + 0.627s}{2.023 + 3.013s + s^2}$$

PID Controller Design Using Reduced Order Model

Consider a reference model

$$M(s) = \frac{4.242s + 25}{s^2 + 7.07s + 25}$$

The equivalent open loop transfer function is

$$\overline{M}(s) = \frac{4.242s + 25}{s^2 + 2.828s}$$

The reduced controller transfer function is

$$R_c(s) = \frac{\overline{M}(s)}{R_2(s)} = \frac{12.285 + 10.641s - 1.236s^2}{s}$$

It is compared by PID controller transfer function

$$= K_1 + \frac{K_2}{s} + K_3s$$

And the value of controller parameters are obtained as

$$K_1 = 10.641, K_2 = 12.285, K_3 = -1.236$$

Thus corresponding closed loop transfer function is

$$R_{CL}(s) = \frac{R_c(s) * R_2(s)}{1 + R_c(s) * R_2(s)} = \frac{5.114s^2 + 23.28s + 17.7}{s^3 + 8.114s^2 + 25.28s + 17.7}$$

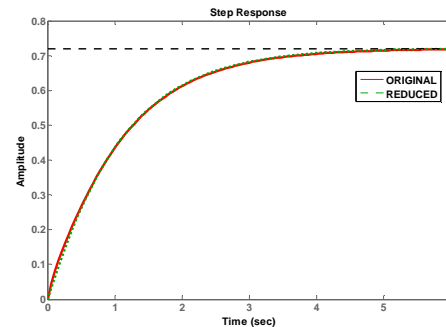


Fig. 1. Comparison of original plant and reduced order system for step response

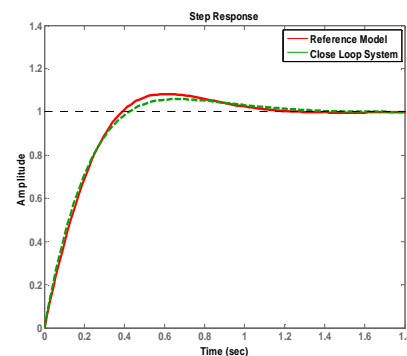


Fig. 2. Comparison of desired reference model and close loop system for step response

TABLE

A Comparison of different parameters of original plant  $G(s)$ , reduced order plant  $R_2(s)$ , and closed loop plant ( $R_{cl}(s)$ )

Sr.No	Parameter	G(s)	R <sub>2</sub> (s)	R <sub>cl</sub> (s)
1	Rise Time(Sec)	2.3149	2.2800	0.2915
2	Settling Time(Sec)	4.0227	4.0028	1.1268
3	Peak Time(Sec)	7.7084	7.7392	0.6601
4	Overshoot	0	0	5.9253
5	Peak	0.7196	0.7214	1.05912

## VI. CONCLUSIONS

In this proposed order reduction method for the linear single-input-single-output high order systems, the determination of denominator polynomial of the reduced model is done by using the cluster method while the numerator coefficients are computed by factor division method. After that PID controller has been design for reduced order model. The proposed algorithm has been explained with a numerical example. A comparison of the step responses of the original and reduced system of second order are shown in the Figure 1 and a comparison of the step responses of the reference model and close loop

reduced order system after including PID controller in loop is shown in figure 2. Table-1 gives the comparison on the basis of various parameters of original plant and second order reduced model and closed loop responses. This method is simple, proficient and takes little computational time.

## REFERENCES

- [1] H. Pade, "Sur la representation approchee d'une fonction par des fractions rationnelles," *Annales Scientifiques de l'Ecole Normale Supieure*, ser. 3, vol. 9, pp. 1-93 (suppl.), 1892.
- [2] Y. Shamash, "Stable reduced-order models using Padé-type approximations," *IEEE Trans. Automat. Contr.* (Tech. Notes and R. J. Schwarz and B. Friedland, *Linear Systems*. New York: McGraw-Hill, 1965.
- [3] G. Parmar, S. Mukherjee, and R. Prasad, System reduction using factor division algorithm and eigen spectrum analysis, *Int. J. Applied Math. Modeling*, Vol. 31, pp. 2542-2552, 2007.
- [4] Lucas T.N., Factor division: A useful algorithm in model reduction *IEE Proc. Pt.D* Vol. 130, No. 6, pp. 362-364, 1980.
- [5] A.K. Sinha, J. Pal, Simulation based reduced order modeling using clustering technique, *Computer and Electrical Engineering*, 16(3), 1990, 159-169.
- [6] J. Pal, A.K. Sinha and N.K. Sinha, Reduced-order modelling using pole clustering and time-moments matching, *Journal of The Institution of Engineers (India)*, Pt EI, 76, 1995, 1-6.
- [7] S. Janardhanan, "Model Order reduction and Controller Design Techniques", 2005
- [8] L. A. Aguirre, "PID tuning based on model matching ", *IEEE electronic letter*, Vol 28, No. 25, pp 2269-2271, 1992.
- [9] H. Lnooka, G. Obinata and M. Takeshima, "Design of digital controller based on series expansion of pulse transfer functions", *Journal of Dynamic systems, measurements and control*, Vol 105, No. 3 pp. 204 -207, 1983.
- [10] A.Varsek, T. Urbacic and B. Filipic, "Genetic Algorithm in controller Design and tuning ", *IEEE transaction on sys. Man and Cyber*, Vol 23, No.5 pp 1330-1339, 1993.
- [11] Z. L. Gaing, "A particle Swarm Optimization approach for optimum design of PID controller in AVR system", *IEEE transaction on energy conversion*, Vol.19 No.2, pp 384-391, 2004.
- [12] M.Gopal, "Control System: Principle and Design" , 3rd Ed., New Delhi, Tata McGraw Hill, 2008.

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**Jasvir Singh Rana** was born in Wajirpur (Saharanpur) U.P, India, in 1976. He received B.Sc. degree from Meerut University, India, in 1997, B.Tech. degree in Electronics & Instrumentation Engineering from V.B.S Purvanchal University, Jaunpur India, in 2001 and M.Tech in Instrumentation Engineering from the School of Instrumentation at Devi Ahilya University,

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